A Primer for Mathematical Economics

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1 Mathematical Foundations

1.1 Preliminaries

1.1.1 Notation

$a, b, \dots$	elements, numbers, variables, parameters, ...	
$\mathbf{a}, \mathbf{b}, \dots$	column vectors	
$\mathcal{A}, \mathcal{B}, \dots$	sets	
$\mathbf{A}, \mathbf{B}, \dots$	matrices	
$a \implies b$	if a, then b (a implies b)	*Implikation*
$a \impliedby b$	if b, then a (a is implied by b)	
$a \iff b$	if and only if a, then b (a is equivalent to b)	*Äquivalenz*
$a :\iff b$	equivalence by definition	
$\{a_1, a_2, \dots, a_n\}$	set of objects (or members, elements)	
$\{a_i\}_{i=1}^{\infty}$	sequence	
(a, b)	ordered pair of elements	
$\{x \mid B(x)\}$	set of all x satisfying $B(x)$	
$\{\,\}, \emptyset$	empty set	
$x \in \mathcal{A}$	x is an element of $\mathcal{A}$ (membership)	
$x \notin \mathcal{A}$	x is not an element of $\mathcal{A}$	
$\subset, \supset, \cap, \cup, \setminus$	operations with sets	☞ *Sec. 7*
$\mathcal{A} \times \mathcal{B}$	cross product, $(a, b) \in \mathcal{A} \times \mathcal{B} \iff a \in \mathcal{A}$ and $b \in \mathcal{B}$	*Kreuzprodukt*
$\forall a$	for all a	*für alle*
$\exists a$	there is an a	*es gibt ein*
$\wedge$	(logical) and	*(logisches) und*
$\vee$	(logical) or	*(logisches) oder*
$\neg$	not (negation)	*Verneinung*
$a = b$	a equals b	*Gleichheit*
$a \coloneqq b$	a is defined to be b	*Definition*
$a \equiv b$	a is identically equal to b	*identisch gleich*

1.1.2 Sets of Numbers

natural numbers	$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$	*natürliche Zahlen*
integers	$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	*ganze Zahlen*
rational numbers	$\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\right\}$	*rationale Zahlen*
real numbers	$\mathbb{R}$ (includes irrational numbers such as π or e) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}, \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}, \ldots, \mathbb{R}^n$ $\mathbb{R}_+$ non-negative real numbers	*reelle Zahlen*
complex numbers	$\mathbb{C}$ $\quad i = \sqrt{-1}$	*komplexe Zahlen*

1.1.3 Greek Alphabet

α	A	alpha	ι	I	iota	ρ, ϱ	P	rho
β	B	beta	κ	K	kappa	σ, ς	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Y	ypsilon
ε, ϵ	E	epsilon	ν	N	nu	ϕ, φ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omikron	ψ	Ψ	psi
θ, ϑ	Θ	theta	π	Π	pi	ω	Ω	omega

1.2 Elementary Arithmetic

1.2.1 Sums and Products

operation	name	a is called	b is called	c is called
$a + b = c$	addition	addend	addend	sum
$a - b = c$	subtraction	minuend	subtrahend	difference
$a \cdot b = c$	multiplication	factor	factor	product
$a : b = c$	division	dividend	divisor	quotient
	fraction $\frac{a}{b}$	numerator	denominator	

commutative law	$a + b = b + a$ and $a \cdot b = b \cdot a$	*Kommutativgesetz*
associative law	$(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	*Assoziativgesetz*
distributive law	$a \cdot (b + c) = a \cdot b + b \cdot c$	*Distributivgesetz*
sum	$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^n a_i$	*Summe*
product	$\sum_{i=1}^n a = n\,a$ $a_1 \cdot a_2 \cdots a_n = \prod_{i=1}^n a_i$	*Produkt*
factor c	$\sum_{i=1}^n c\,a_i = c \sum_{i=1}^n a_i$	*Faktor*
sum of sums	$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$	

1.2.2 Fractions

Brüche

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{cb}$$

$$\frac{\textit{Zähler}}{\textit{Nenner}}$$

$$\frac{a}{b} = \frac{c}{d} \iff a \cdot d = c \cdot b$$

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}, \qquad \frac{a : c}{b : c} = \frac{a}{b}, \qquad \frac{a+c}{b+c} \neq \frac{a}{b} \quad (c \neq 0)$$

1.2.3 Powers and (Square) Roots

power (a^n is called the n-th power of a, a = base, n = exponent) *Potenz*

$$a^0 = 1 \;\; (a \neq 0), \quad a^1 = a, \quad a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad \ldots \; a^n$$

root $\sqrt[n]{a} = x \iff x^n = a \qquad (a \geq 0,\; n \in \mathbb{N},\; n \geq 2, x \geq 0)$ *Wurzel*

square root $\sqrt[2]{a} = \sqrt{a}$ *Quadratwurzel*

general rules

$$a^n \cdot a^m = a^{n+m} \qquad a^n : a^m = a^{n-m} \quad (a \neq 0)$$
$$a^n \cdot b^n = (a \cdot b)^n \qquad a^n : b^n = (a : b)^n \quad (b \neq 0)$$
$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0) \qquad a^{1/n} = \sqrt[n]{a} \quad (n \neq 0)$$
$$(a^n)^m = (a^m)^n = a^{n \cdot m} \qquad a^{m/n} = \sqrt[n]{a^m} \quad (n \neq 0)$$

1.2.4 Logarithms

Logarithmen

$\log_b a$ is the logarithm of a to the basis b.

$$x = \log_b a \iff b^x = a \qquad (a, b > 0 \quad \text{and} \quad b \neq 1)$$

rules $\log(a \cdot b) = \log a + \log b, \quad \log(a : b) = \log a - \log b$

$\log(a^b) = b \log a$

$\log_c a = \log_c b \cdot \log_b a$

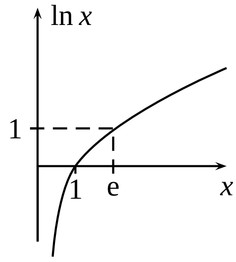
special cases $\log_b b = 1$ and $\log_b 1 = 0$

$e \approx 2.718\,282$

natural logarithm $\log_e a = \ln a$

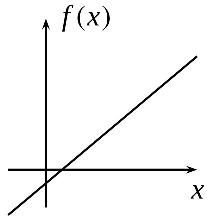
special cases $\ln 1 = 0, \quad \ln e = 1, \quad \ln e^a = a, \quad e^{\ln a} = a$

$$\log_b a = \frac{\ln a}{\ln b}$$



1.3 Equations

1.3.1 Linear Equations



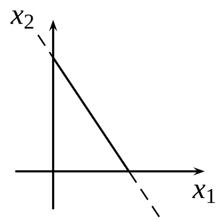
normal form of a linear equation

$$a + b\,x = 0 \implies \begin{cases} x = -a/b & \text{if } b \neq 0 \\ \text{no solution} & \text{if } b = 0 \quad \text{and} \quad a \neq 0 \\ x \text{ arbitrary} & \text{if } b = 0 \quad \text{and} \quad a = 0 \end{cases}$$

linear equation for more than one variable, i.e. $x_1, x_2, \ldots, x_n$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

☞ *Sec. 5.3*



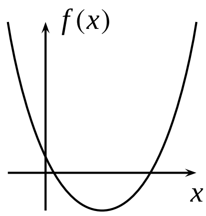
Examples *The consumption expenditure e for two amounts of goods x_1, x_2 at given commodity prices p_1, p_2 results from*

$$e = p_1x_1 + p_2x_2 \iff x_2 = \frac{e}{p_2} - \frac{p_1}{p_2}\,x_1\,.$$

The factor cost c for two amounts of factors of production v_1, v_2 at given factor prices q_1, q_2 are given by

$$c = q_1v_1 + q_2v_2 \iff v_2 = \frac{c}{q_2} - \frac{q_1}{q_2}\,v_1\,.$$

1.3.2 Quadratic Equations



normal form of a quadratic equation

$$x^2 + ax + b = 0 \implies x_{1,2} = -a/2 \pm \sqrt{(a/2)^2 - b}$$

special cases for $a = 0$, $b = 0$, and $a^2 = 4b$; no real solution for $a^2 < 4b$;

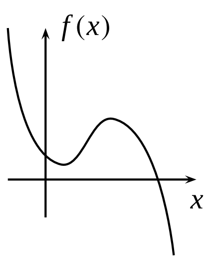
$$0 = cd - (c+d)x + x^2 = (c-x)(d-x) \implies x_1 = c,\; x_2 = d$$

1.3.3 Cubic Equations

cubic equation $\quad ax^3 + bx^2 + cx + d = 0$

replace x by $x - b/3a$ to obtain $\quad x^3 + px + q = 0$

Cardano's formulas



$$x_1 = u + v, \qquad x_{2,3} = -\frac{u+v}{2}\sqrt{3} \pm \frac{u-v}{2}\sqrt{-3},$$

$$u = \left(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{w}{27}}\right)^{1/3} \qquad v = \left(-\frac{q}{2} - \frac{1}{2}\sqrt{\frac{w}{27}}\right)^{1/3}$$

$$w = 4p^3 + 27q^2$$

$w < 0$: three different real roots x_1, x_2, x_3

$w = 0$: three real roots, at least two roots are equal

$w > 0$: one real root and two complex roots

1.3.4 Polynomials

Polynome

polynomial of degree n ($a_n \neq 0$)

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

Fundamental Theorem of Algebra There are n (not necessarily distinct, real or complex) constants $x_1, \ldots, x_n$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_n (x - x_1) \cdots (x - x_n)$$

1.3.5 Further Equations

power	$x^a = b \iff x = b^{1/a}$	$(a \neq 0)$
exponential	$a^x = b \iff x = \dfrac{\ln b}{\ln a}$	$(a > 0, a \neq 1, b > 0)$
logarithm	$\ln x = b \iff x = \mathrm{e}^b$	

1.4 Inequalities

Ungleichungen

$$a < b \iff b > a \iff b - a > 0$$
$$a < b \quad \text{and} \quad a < c \implies a < c$$
$$a < b \implies a + c < b + c \quad \text{and} \quad a \cdot c < b \cdot c \quad (c > 0)$$
$$a < b \implies -a > -b \quad \text{and} \quad 1/a > 1/b \quad (a, b \neq 0)$$
$$a < b \quad \text{and} \quad c < d \implies a + c < b + d$$
$$a < b,\ b > 0,\ \text{and } 0 < c < d \implies a \cdot c < b \cdot d$$

notation $a \leq b \iff (a < b \text{ or } a = b)$

intervals

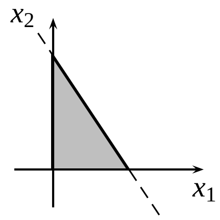
Intervalle

open interval from a to b	$\{x \mid a < x < b\} = (a, b)$
closed interval from a to b	$\{x \mid a \leq x \leq b\} = [a, b]$
half-open interval from a to b	$\{x \mid a < x \leq b\} = (a, b]$
half-open interval from a to b	$\{x \mid a \leq x < b\} = [a, b)$
real numbers	$\{x \mid -\infty < x < \infty\} = (-\infty, \infty) = \mathbb{R}$
non-negative real numbers	$\{x \mid 0 \leq x < \infty\} = [0, \infty) = \mathbb{R}_+$

Example *The consumption expenditure c must not exceed the income y, that is $y \geq c$. This budget constraint can be rewritten as*

$$y \geq p_1 x_1 + p_2 x_2 \iff x_2 \leq \frac{y}{p_2} - \frac{p_1}{p_2} x_1 .$$

The budget constraint together with non-negative amounts of the two commodities ($x_1 \geq 0$, $x_2 \geq 0$) yields the feasible set. It consists of all commodity bundles that can be bought from the income y. ☞ *Sec. 6.1*



1.5 Mean Values

Mittelwerte

arithmetic mean	$m_a = \frac{a+b}{2}$	$m_a = \frac{a_1 + a_2 + \cdots + a_n}{n}$	*arithmetisches Mittel*
weighted mean	$m_w = w_1 a + w_2 b$	$(w_1 + w_2 = 1,\ w_1, w_2 \geq 0)$	*gewichtetes Mittel*
geometric mean	$m_g = \sqrt{a\,b}$	$m_g = \sqrt[n]{a_1 \cdot a_2 \cdots a_n}$	*geometrisches Mittel*
harmonic mean	$m_h = \frac{2}{\frac{1}{a} + \frac{1}{b}}$	$m_h = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$	*harmonisches Mittel*

Example *price index (base year* 0*, current year t), commodity prices (*p_1, p_2*)*

Paasche $P = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^0 x_1^t + p_2^0 x_2^t}$ *commodity bundle* (x_1^t, x_2^t)

Laspeyres $P = \frac{p_1^t x_1^0 + p_2^t x_2^0}{p_1^0 x_1^0 + p_2^0 x_2^0}$ *commodity bundle* (x_1^0, x_2^0)

2 Financial Mathematics

2.1 Sequences and Series

2.1.1 Properties of Sequences

finite sequence	$\{a_1, a_2, a_3, \ldots, a_n\} \iff \{a_i\}_{i=1}^n$	*endliche Folge*
sequence	$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \iff \{a_i\}_{i=1}^\infty \iff \{a_i\}$	*Folge*
series	$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^\infty a_i$	*Reihe*
arithmetic sequence	$\{a_i\}$ with $a_{i+1} - a_i = d =$ const. for all i	
geometric sequence	$\{a_i\}$ with $a_{i+1}/a_i = q =$ const. for all i	

Boundedness A sequence $\{a_i\}$ is said to be *Beschränktheit*

- bounded if $|a_i| \leq K$ for every i
- bounded from below if a lower bound K exists such that $a_i \geq K$ for all i
- bounded from above if an upper bound K exists such that $a_i \leq K$ for all i

The *greatest lower bound* is referred to as *infimum* and the *least upper bound* is called *supremum*. It is perfectly possible for a supremum $\overline{K}$ and an infimum $\underline{K}$ that $\underline{K} < a_i < \overline{K}$ for all i. *Infimum* *Supremum*

Monotonicity A sequence $\{a_i\}$ is said to be *Monotonie*

- monotone increasing if $a_{i+1} \geq a_i$ for every i
- monotone decreasing if $a_{i+1} \leq a_i$ for every i

Replace $\geq$ and $\leq$ by $>$ and $<$ to obtain strict monotonicity. *strenge Monotonie*

Limit The real number r is the *limit* of the sequence $\{x_i\}$ if for *any* positive ε, there is a number N such that for all $i \geq N$, we have $|x_i - r| < \varepsilon$. *Grenzwert*

Convergence We say that the sequence *converges* to r *Konvergenz*

$$\lim_{i\to\infty} \{x_i\} = r \quad \text{or, simply,} \quad x_i \to r.$$

Sequences can have at the most one limit.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\right\} \to 0; \qquad \left\{+1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4} \ldots\right\} \to 0$$

$$\left\{0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8} \ldots\right\} \to 0; \qquad \lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = \mathrm{e}^k$$

general rules

$$\begin{aligned} x_i \to x, &\implies a\,x_i \to a\,x \\ x_i \to x, &\implies a + x_i \to a + x \\ x_i \to x,\ y_i \to y &\implies x_i \pm y_i \to x \pm y \\ x_i \to x,\ y_i \to y &\implies x_i\,y_i \to x\,y \end{aligned}$$

2.1.2 Arithmetic Series

arithmetische Reihen

In an arithmetic sequence $\{a_i\}$ the distance between any two adjacent members is constant.

difference $d = a_n - a_{n-1} = \cdots = a_3 - a_2 = a_2 - a_1 = \text{const.}$

nth member $a_n = a_1 + (n-1)d$

nth partial sum $s_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

sum $s_n = \frac{n}{2}(a_1 + a_n)$

arithmetic series $s = \sum_{i=1}^{\infty} a_i$

2.1.3 Geometric Series

geometrische Reihen

In a geometric sequence $\{a_i\}$ the quotient of any two subsequent members is constant.

quotient $q = \frac{a_n}{a_{n-1}} = \cdots = \frac{a_3}{a_2} = \frac{a_2}{a_1} = \text{const.}$

nth member $a_n = a\,q^{n-1}$

nth partial sum $s_n = a + aq + aq^2 + \cdots + aq^{n-2} + aq^{n-1}$

sum $s_n = a\,\frac{q^n - 1}{q - 1} \quad (q \neq 1)$

geometric series $s = \lim_{n\to\infty} s_n = a + aq + aq^2 + aq^3 + \cdots = \frac{a}{1-q} \quad (|q| < 1)$

2.1.4 Power Series

Potenzreihen

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{1}{2}\, n(n+1)$$

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6}\, n(n+1)(2n+1)$$

$$1^3 + 2^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = \frac{1}{4}\, n^2(n+1)^2$$

$$1^4 + 2^4 + \cdots + n^4 = \sum_{i=1}^{n} i^4 = \frac{1}{30}\, n(n+1)(2n+1)(3n^2+3n-1)$$

$$\sum_{i=0}^{\infty} \frac{n^i}{i!} = \mathrm{e}^n$$

2.2 Depreciation

Abschreibung

(1) constant depreciation

depreciation $a_t = a = \dfrac{K_0 - K_T}{T} \qquad (t = 1, \ldots, T)$

arithmetic sequence $a_t - a_{t-1} = 0 = \text{const.}$

cumulated depreciation $s_t = \dfrac{t}{2}(a_1 + a_t) = t\,a$

asset value after t years $K_t = K_{t-1} - a = K_0 - t\,a$

Example *Asset value at acquisition cost $K_0 = 8000$ €. Book value $K_T = 1500$ € after the period of depreciation of $T = 5$ years. Constant depreciation per year:* $a = (8000 - 1500)/5 = 1300$ €.

Anschaffungswert
Nutzungsdauer

(2) arithmetic-degressive depreciation

special case with $D = \dfrac{2(K_0 - K_T)}{T(T+1)}$ such that $a_T = D$

depreciation $a_t = (T - t + 1)D$

digitale Abschreibung

arithmetic sequence $a_t - a_{t-1} = -D = \text{const.}$

cumulated depreciation $s_t = \dfrac{t}{2}(a_1 + a_t) = \dfrac{t}{2}(2T - t + 1)D$

asset value after t years $K_t = K_{t-1} - a_t = K_0 - s_t$

Example *Same asset as before.* $D = 2(8000 - 1500)/(5(5+1)) = 433.333.$

Year:	*0*	*1*	*2*	*3*	*4*	*5*
Depreciation:		*2166.7*	*1733.3*	*1300.0*	*866.7*	*433.3*
Book value:	*8000.0*	*5833.3*	*4100.0*	*2800.0*	*1933.3*	*1500.0*

(3) geometric-degressive depreciation

depreciation $a_t = D\,K_{t-1} = D(1-D)^{t-1}K_0$

geometric sequence $q = a_t/a_{t-1} = K_t/K_{t-1} = 1 - D = \text{const.}$

cumulated depreciation $s_t = a_1 \dfrac{q^t - 1}{q - 1} = K_0\,[1 - (1-D)^t]$

asset value after t years $K_t = K_{t-1} - a_t = K_0 - s_t = (1-D)^t K_0$

Example *Same asset as before.* $1500 = (1-D)^5\,8000 \implies D \approx 28.45\%$.

Year:	*0*	*1*	*2*	*3*	*4*	*5*
Depreciation:		*2276.1*	*1628.5*	*1165.2*	*833.7*	*596.5*
Book value:	*8000.0*	*5723.9*	*4095.4*	*2930.2*	*2096.5*	*1500.0*

2.3 Interest and Compound Interest

compound interest = Zinseszinsen

2.3.1 Discrete Payments of Interest

interest rate r $5\% = 0.05$ *Zinssatz*

interest factor $q = 1 + r$ *Zinsfaktor*

interest rK_0 $(-rK_0 = \text{discount})$ *Zinsen (Rabatt)*

$K_1 = q\,K_0 = K_0 + rK_0$

accumulated value $K_n = q^n\,K_0 = (1+r)^n\,K_0$ *Endwert*

present (discounted) value $K_0 = q^{-n}\,K_n = \dfrac{1}{(1+r)^n}\,K_n$ *Barwert*

growth rate $g(K) = \dfrac{K_n - K_{n-1}}{K_{n-1}} = \dfrac{q^n - q^{n-1}}{q^{n-1}} = q - 1 = r$ *Wachstumsrate*

2.3.2 Shorter Periods of Interest Payments

unterjährige Verzinsung

interest rate r per year; interest rate r_m per month ($m = 12$), per week ($m = 52$), or per day ($m = 365$ or 360); m = interest periods *Zinsperioden*

equivalent interest rates $1 + r = (1 + r_m)^m$ or $r = (1 + r_m)^m - 1$

accumulated value $K_{m\cdot n} = (1 + r_m)^{m\cdot n}\,K_0$ (n = years)

effective yearly rate $\tilde{r} = (1 + r/m)^m - 1$

1% per month ≈ 12.68% per year
1% discount if you pay within one week ≈ 67.8% per year
3% discount if you pay within one week ≈ 365.1% per year

approximation $r_m = r/m$, then

$$1 + r = (1 + r_m)^m \overset{!}{=} (1 + r/m)^m \xrightarrow{x = m/r} (1 + 1/x)^{x\,r}$$

An ever increasing number of payments within each year $m \to \infty$ implies $x = m/r \to \infty$ for fixed r

$$\lim_{x\to\infty} (1+1/x)^x \to \mathrm{e}$$

For m, and x, sufficiently large: $(1+r)^n = (1+r_m)^{mn} = \mathrm{e}^{rn}$

$1.1^{10} = 2.5937$
$\approx 2.7183 = \mathrm{e}^{0.1\cdot 10}$

2.3.3 Continuous Compounding of Interest

accumulated value $K(t) = K(0)\,\mathrm{e}^{rt}$ — *Endwert*

present value $K(0) = K(t)\,\mathrm{e}^{-rt}$ — *Barwert*

growth rate $g(K)$ $\dot{K}(t) \equiv \frac{\mathrm{d}K(t)}{\mathrm{d}t} = K(0)\,r\,\mathrm{e}^{rt}, \quad g(K) \equiv \frac{\dot{K}(t)}{K(t)} = r$ — *Wachstumsrate*

2.3.4 Internal Rate of Return

interner Zinssatz

present value $K_0 = R_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} = \sum_{i=0}^{n} \frac{R_i}{(1+r)^i}$

The internal rate of return is the smallest positive number r which solves

$$R_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} = 0.$$

2.4 Annuities

2.4.1 Annuity Immediate

An *annuity* denotes a sum of money – say a – paid at regular intervals (as every year). *Perpetuis* are assets that last forever (e.g., land) and pay a € each year from now to eternity ($n \to \infty$).

Rente, a = Rate
ewige Rente

An *annuity immediate*, or *ordinary annuity*, is paid at the end of each period. — *nachschüssige Rente*

☞ *Sec. 2.1.3*

accumulated value $AV_n = a\,\frac{q^n - 1}{q - 1}$ — $AV_n = q^n\,PV_n$

present value $PV_n = a\,\frac{1}{q^n}\,\frac{q^n - 1}{q - 1} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$

Dissect each year into m periods of equal length. Payments a per period (e.g., month) are paid the interest rate r p.a. but no compound interest. The number of "valid" subperiods within a period is $(m-1)m/2$.

p.a. = per annum
= per year

interest payment for the first year $a\,(m-1)m/2 \cdot r/m$

accumulated value $AV_1 = a\left[m + \frac{1}{2}\,(m-1)\,r\right]$ — *jährliche Ersatzrente*

accumulated value $AV_n = AV_1\,\frac{q^n - 1}{q - 1}$

An *annuity* can also be defined as an asset PV_n that pays a fixed sum a each year for a specified number of years, say n.

present value factor $PVF = \frac{q^n - 1}{q^n(q-1)} \implies PV_n = a \cdot PVF$

pay back factor $PBF = \frac{q^n(q-1)}{q^n - 1} \implies a = PV_n \cdot PBF$

2.4.2 Annuity Due

An *annuity due* is paid at the beginning of each period. *vorschüssige Rente*

accumulated value $AV_n = a\,q\,\frac{q^n - 1}{q - 1}$

☞ Sec. 2.1.3
with $a = A\,q$

present value $PV_n = a\,\frac{1}{q^{n-1}}\,\frac{q^n-1}{q-1} = \frac{a\,(1+r)}{r}\left[1 - \frac{1}{(1+r)^n}\right]$ $AV_n = q^n\,PV_n$

Dissect each year into m periods of equal length. Payments a per period (e.g., month) are paid the interest rate r p.a. but no compound interest. The number of "valid" subperiods within a period is $(m+1)m/2$.

p.a. = per annum
= per year

interest payment for the last year $a\,(m+1)m/2 \cdot r/m$

accumulated value $AV_1 = a\left[m + \frac{1}{2}\,(m+1)\,r\right]$

accumulated value $AV_n = AV_1\,\frac{q^n - 1}{q - 1}$

2.5 Redemption

Tilgung

initial dept K_0, repayment period N years, interest rate r, ordinary payments at the end of each interest period, interest payment $Z_n = rK_{n-1}$
(1) constant repayment rate *Annuitätentilgung*

repayment rate $A = K_0\,\frac{q^N(q-1)}{q^N - 1}$ $(A = R_n + Z_n)$ *Annuität*

redemption installment $R_n = (A - r\,K_0)\,q^{n-1}$ *Tilgungsrate*

remaining dept $K_n = K_0\,q^n - A\,\frac{q^n - 1}{q - 1} = K_0\,\frac{q^N - q^n}{q^N - 1}$ *Restschuld*

m redemption periods per interest period and annual interest payment *m Tilgungsperioden je Zinsperiode*

redemption installment per period $a = \frac{A}{m + \frac{1}{2}(m-1)r}$

(2) constant redemption installment *Ratentilgung*

redemption installment $R = K_0/N$ *Tilgungsrate*

remaining dept $K_n = K_0 - nR$ *Restschuld*

interest payment $Z_n = rK_{n-1} = rK_0\left[1 - \frac{1}{N}\,(n-1)\right]$

repayment rate $A_n = R + Z_n$ — *Annuität*

m redemption periods per interest period and annual interest payment — *m Tilgungsperioden je Zinsperiode*

redemption installment $R = K_0/(mN)$ — *Tilgungsrate*

interest for the n^{th} period $Z_n = rK_0 \left[1 - \frac{1}{mN}\left(nm - \frac{1}{2}(m+1)\right)\right]$

2.6 Amortization

Amortisation

The amortization period denotes that point of time t (t is not necessarily integer) at which the asset value K_0 equals the present value PV_t of an annuity A.

annuity immediate $K_0 = A\,\frac{q^t - 1}{q^t\,(q-1)} \iff t = \log_q\left(\frac{A}{A - (q-1)\,K_0}\right)$

annuity due $K_0 = A\,\frac{q^t - 1}{q^{t-1}\,(q-1)} \iff t = \log_q\left(\frac{qA}{qA - (q-1)\,K_0}\right)$

Example *Suppose you buy an asset at acquisition cost $K_0 = 8000$ € in order to receive an annuity of $A = 500$ €. If you claim an interest rate of $r = 5\%$, or $q = 1.05$, then it takes a period of*

$$t = \ln\left(\frac{500}{500 - 0.05 \cdot 8000}\right) / \ln 1.05 \approx 33.0 \qquad \text{(annuity immediate)}$$

$$t = \ln\left(\frac{1.05 \cdot 500}{1.05 \cdot 500 - 0.05 \cdot 8000}\right) / \ln 1.05 \approx 29.4 \qquad \text{(annuity due)}$$

years until the asset is payed back.

3 Calculus of One Variable

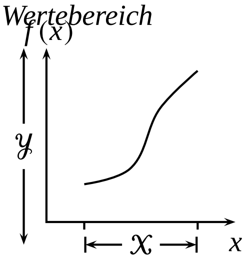
3.1 Functions on $\mathbb{R}^1$

3.1.1 Functions and Graphs of Functions

Functions A function is a mapping $f\colon \mathcal{X} \to \mathcal{Y}$ where each element of the *domain* $\mathcal{X}$ is uniquely assigned to one element of the *target* $\mathcal{Y}$.

Funktion
Definitionsbereich
Wertebereich

$$f\colon x \mapsto y = f(x), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}$$



Variables The argument x of the function f is called the *independent variable*, while the functional value y is referred to as *dependent variable*.
The variables x, y are also said to be *endogenous* in order to distinguish them from *exogenous variables*, or *parameters*, the values of which are given by the environment (e.g., prices in competitive markets).

$$y = a + bx \qquad \text{variables } y, x; \quad \text{parameters } a, b$$

Graphs The collection of all pairs with coordinates $(x, f(x))$ gives a graphical representation of the function f, i.e. the graph of f.

Inverse If a function $g\colon y \mapsto g(y) = x$ with $g\colon \mathcal{Y} \to \mathcal{X}$ exists, then g is called the *inverse* of $f\colon x \mapsto f(x) = y$ with $f\colon \mathcal{X} \to \mathcal{Y}$.

Inverse

Examples

(1) $y = ax + b \iff x = \dfrac{y}{a} - \dfrac{b}{a} \quad (a \neq 0)$
(2) $y = x^2$ *with* $\mathcal{X} = \mathbb{R}$ *is not invertible, but for* $\mathcal{X} = \mathbb{R}_+$ *we obtain* $x = \sqrt{y}$.

3.1.2 Properties of Functions

Monotonicity The function f is said to be

Monotonie

monotone increasing	if for all $x_1, x_2,\ x_1 < x_2\colon\ f(x_1) \leq f(x_2)$
monotone decreasing	if for all $x_1, x_2,\ x_1 < x_2\colon\ f(x_1) \geq f(x_2)$

Replace $\leq$ and $\geq$ by $<$ and $>$ for *strict monotonicity*.

strenge Monotonie

Limits The function f has the limit y_0 at x_0 if *each* sequence $\{x_i\}$ with $x_i \to x_0$ and $x_n \neq x_0$ implies a sequence $\{f(x_i)\}$ which converges to y_0.

Grenzwerte

$$\lim_{x \to x_0} f(x) = y_0$$

General rules. If $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, then

☞ l'Hôpital's rule

$$\lim_{x\to a}[f(x) \pm g(x)] = F \pm G$$
$$\lim_{x\to a}[f(x)\, g(x)] = F\, G$$
$$\lim_{x\to a}[f(x)/g(x)] = F/G \qquad (G \neq 0)$$
$$\lim_{x\to a}[f(x)]^{p/q} = F^{p/q} \qquad (F^{p/q} \text{ is defined})$$

Continuity The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, if whenever $\{x_i\}$ is a sequence which converges to x_0, then the sequence $\{f(x_i)\}$ converges to $f(x_0)$.

Stetigkeit

Let f and g be functions which are continuous at x. Then, $f+g$, $f-g$, $f\cdot g$, and f/g $(g(x) \neq 0)$ are all continuous at x.

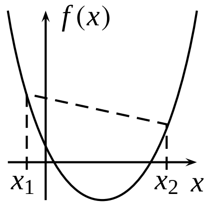
Additivity The function f is said to be *additive* if

Additivität

$$f(x_1+x_2) = f(x_1) + f(x_2) \quad \forall\, x_1, x_2$$

Convexity The function f is said to be *convex* if

Konvexität



$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda f(x_2) \quad \forall\, x_1, x_2, \quad \forall \lambda \in (0,1)$$

The function is *concave* if $\leq$ is replaced by $\geq$.
Strict convexity and strict concavity refer to strict inequality, that is $<$ and $>$.
Each convex function $f\colon \mathcal{X} \to \mathbb{R}$ is continuous on $\mathcal{X}$.

Linearity The function f is said to be *linear* if

Linearität

$$f(x_1+x_2) = f(x_1) + f(x_2) \quad \text{and} \quad \lambda\, f(x) = f(\lambda\, x) \quad \forall\, \lambda$$

Extreme points The function $f\colon \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X} = [a,b]$ has

Extrema
Maximum
Minimum

an absolute maximum at x_0 if	$f(x_0) > f(x)$	$\forall\, x \in \mathcal{X}\ \ (x \neq x_0)$
an absolute minimum at x_0 if	$f(x_0) < f(x)$	$\forall\, x \in \mathcal{X}\ \ (x \neq x_0)$
a relative maximum at x_0 if	$f(x_0) > f(x)$	$\forall\, x \in U_{x_0}\ \ (x \neq x_0)$
a relative minimum at x_0 if	$f(x_0) < f(x)$	$\forall\, x \in U_{x_0}\ \ (x \neq x_0)$

where $U_{x_0} = \{x \mid x_0 - \varepsilon < x_0 < x_0 + \varepsilon,\ \varepsilon > 0\}$ is an ε-ball about x_0 for an arbitrarily small but positive ε.

ε-Umgebung

3.1.3 Important Classes of Function

linear function	$f(x) = a + b\,x$
quadratic function	$f(x) = a_0 + a_1\,x + a_2\,x^2$
cubic function	$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3$
polynomial	$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
square root	$f(x) = \sqrt{x}$
exponential function	$f(x) = a^x$
logarithmic function	$f(x) = \ln x$
explicit function	$y = f(x)$
implicit function	$0 = g(y,x)$

Examples

(1) utility function $\quad U = x_1^a x_2^b \quad$ *with* $\quad U = const.$

$$\Longrightarrow \quad g(x_1,x_2) = 0 = x_1^a x_2^b - U$$

indifference curve $\quad \Longleftrightarrow \quad x_2 = \left(U\, x_1^{-a}\right)^{1/b} \qquad (b \neq 0,\, x_1 > 0,\, U > 0)$

(2) production function $x = v_1^a v_2^b$ *with* $x = const.$

$$\Longrightarrow g(v_1, v_2) = 0 = v_1^a v_2^b - x$$

isoquant $\Longleftrightarrow v_2 = \left(x\, v_1^{-a}\right)^{1/b} \qquad (b \neq 0,\, v_1 > 0,\, x > 0)$

(3) circle with center (x_0, y_0) *and radius* $r = const.$ — *Kreis*

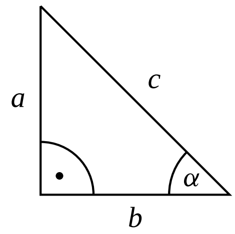
$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

composed forms — *zusammengesetzte Funktionen*

$$f(x) + g(x), \quad f(x) - g(x), \quad f(x) \cdot g(x), \quad \frac{f(x)}{g(x)}, \quad g(x)^{f(x)}, \ldots$$

trigonometric functions — *Trigonometrische Funktionen*
Pythagoras' theorem for right-angled triangles: $a^2 + b^2 = c^2$
a, b = legs of the triangle, c = hypotenuse

a = Gegenkathete
b = Ankathete
c = Hypotenuse



360° = 2π radians
$\pi \approx 3.141\,593$

The sine, cosine, tangent, and cotangent functions are defined by

$$\sin\alpha = \frac{a}{c}, \qquad \cos\alpha = \frac{b}{c}, \qquad \tan\alpha = \frac{a}{b}, \qquad \cot\alpha = \frac{b}{a}$$

	0 0°	π/6 30°	π/4 45°	π/3 60°	π/2 90°
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0

$$\sin^2\alpha + \cos^2\alpha = 1, \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \tan\alpha = \frac{1}{\cot\alpha},$$

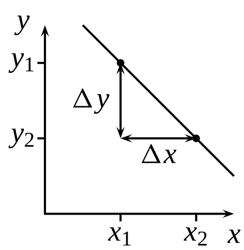
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(180^\circ \pm \alpha) = \pm\tan\alpha$$

3.2 Differentiation

3.2.1 Slopes of Curves



The slope of a linear function f follows from

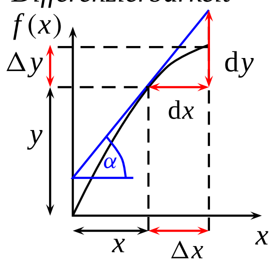
$$y = f(x) = a + bx \Longrightarrow \tan\alpha = b = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

difference quotient — *Differenzenquotient*

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiability A function f is differentiable at x_0 if the following limit exists.

Differenzierbarkeit



$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \quad \text{or} \quad \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

derivative f'

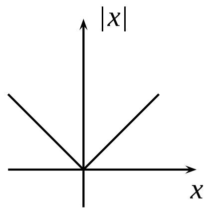
approximation $f(x+\mathrm{d}x) \approx f(x) + \mathrm{d}x\, f'(x)$ (dx small)
☞ Taylor's formula (Sec. 3.2.5)

differential quotient $\dfrac{\mathrm{d}\,f(x)}{\mathrm{d}x} = f'(x)$ — *Differentialqotient*

differential $\mathrm{d}\,f(x) = f'(x)\mathrm{d}x$ — *Differential*

slope $f'(x) \gtrless 0 \iff$ curve (function) increases / decreases — *Steigung*

$f'(x) = 0 \iff$ curve has a horizontal tangent at x



The function $f(x) = |x|$ is not differentiable at $x_0 = 0$. The one-sided limits

$$\lim_{x\uparrow 0} \frac{|x|-0}{x-0} = -1 \quad \text{and} \quad \lim_{x\downarrow 0} \frac{|x|-0}{x-0} = 1$$

exist, but they are different.

3.2.2 General Rules of Differentiation

power rule	$f(x) = x^n$	$\Longrightarrow$	$f'(x) = n\,x^{n-1}$
sum	$F(x) = f(x) + g(x)$	$\Longrightarrow$	$F'(x) = f'(x) + g'(x)$
difference	$F(x) = f(x) - g(x)$	$\Longrightarrow$	$F'(x) = f'(x) - g'(x)$
product	$F(x) = f(x)\,g(x)$	$\Longrightarrow$	$F'(x) = f'(x)\,g(x) + f(x)\,g'(x)$
quotient	$F(x) = \dfrac{f(x)}{g(x)}$	$\Longrightarrow$	$F'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$

Potenzregel

$g(x) \neq 0$

differentiation of the inverse g of f (i.e., $y = f(x) \iff x = g(y)$)

$$y_0 = f(x_0),\ f'(x_0) \neq 0: \qquad g'(y_0) = \frac{1}{f'(x_0)}$$

Examples *price-demand function* $p(x) = a - b\,x$, $p'(x) = -b$
revenue $r(x) = p(x)\,x = a\,x - b\,x^2$,
marginal revenue: $r'(x) = p'(x)x + p(x) = \left(\frac{1}{\eta_{xp}} + 1\right)\, p(x) = a - 2bx$

☞ *Sec. 3.3.4*

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin x$	$\cos x$	e^x	e^x
x^n	nx^{n-1}	$\cos x$	$-\sin x$	a^x	$a^x \ln a$
$\sqrt{x}$	$\frac{1}{2}x^{-1/2}$	$\tan x$	$\dfrac{1}{\cos^2 x}$	$\ln x$	$1/x$
$c\,g(x)$	$c\,g'(x)$	$\cot x$	$-\dfrac{1}{\sin^2 x}$	$\log_a x$	$\dfrac{1}{x \ln a}$

derivatives
= Ableitungen

3.2.3 Chain Rule

Kettenregel

notation
$f(g(x)) \equiv (f \circ g)(x)$

$$F(x) = f(g(x)) \implies F'(x) = f'(g(x)) \cdot g'(x)$$
$$f(x) = \ln g(x) \implies f'(x) = \frac{g'(x)}{g(x)}$$

Example (isoquant from p. 14)

$$v_2(v_1) = \left(x\, v_1^{-a}\right)^{1/b} \qquad (b \neq 0,\, v_1 > 0,\, x > 0)$$
$$v_2'(v_1) = \frac{1}{b}\left(x\, v_1^{-a}\right)^{1/b-1} x\,(-a) v_1^{-a-1} = -\frac{a}{b}\, x^{1/b}\, v_1^{-a/b-1}$$

3.2.4 Higher Order Derivatives

$$f''(x) = (f'(x))', \qquad f^{(n)} = (f^{(n-1)}(x))' \qquad (n > 1)$$

Example
$f(x) = x^4,\ f'(x) = 4x^3,\ f''(x) = 12x^2,\ f'''(x) = 24x,\ f^{(4)}(x) = 24$

Convexity A function f with derivative f' is (strictly) convex in an interval (a, b) if and only if f' is (strictly) monotone increasing in (a, b).

Konvexität

$$f''(x) \geq 0 \quad \forall\, x \in (a, b) \implies f \text{ is convex in } (a, b)$$
$$f''(x) \leq 0 \quad \forall\, x \in (a, b) \implies f \text{ is concave in } (a, b)$$

Replace $\geq$ and $\leq$ by $>$ and $<$ for strict convexity and strict concavity.

3.2.5 Taylor's Formula

n^{th} order Taylor polynomial approximation of $f(x_0 + h)$

$$f(x_0 + h) \approx f(x_0) + \frac{f'(x_0)}{1!}\, h + \frac{f''(x_0)}{2!}\, h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\, h^n$$

☞ *Sec. 8.1*

Lagrangean remainder R_n (approximation error)

Restglied von Lagrange

$$R_n(x_0 + h) = \frac{f^{(n+1)}(c)}{(n+1)!}\, h^{n+1} \qquad \text{for some } x_0 \leq c \leq x_0 + h$$

Example $f(x) = x^2$
(1) $f(x_0 + h) \approx f(x_0) + f'(x_0)\, h, \quad R_1(x_0 + h) = \frac{1}{2!}\, f''(c)\, h^2$
$\implies (x_0 + h)^2 = x_0^2 + 2hx_0 + h^2 \approx x_0^2 + 2hx_0, \quad R_1(x_0 + h) = h^2$
(2) $f(x_0 + h) \approx f(x_0) + f'(x_0)\, h + \frac{1}{2}\, f''(x_0) h^2, \quad R_2(x_0 + h) = \frac{1}{3!}\, f'''(c)\, h^3$
$\implies (x_0 + h)^2 = x_0^2 + 2hx_0 + h^2, \quad R_2(x_0 + h) = 0$

Taylor series

Taylor Reihen

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + - \cdots \qquad |x| < 1$$

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \cdots \qquad |x| < 1$$

☞ *Sec. 8.1*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots \qquad -1 < x \le 1$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots$$

3.2.6 Indeterminate Forms

The limit of the qotient $f(x)/g(x)$ as $x \to a$ yields frequently indetermined forms such as $0/0$ or $\pm\infty/\pm\infty$. If f and g are differentiable, *l'Hôpital's rule* states that

Regel von l'Hôpital

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The rule is also valid for one-sided limits $x \uparrow a$, $x \downarrow a$, $x \to \infty$ and $x \to -\infty$.

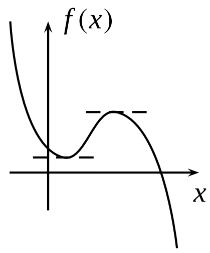
Examples

"$-\infty/\infty$" $$\lim_{x \to \infty} \frac{1-3x^2}{5x^2+x-1} = \lim_{x \to \infty} \frac{-6x}{10x+1} = \lim_{x \to \infty} \frac{-6}{10} = \frac{-3}{5}$$

"$0/0$" $$\lim_{x \to 4} \frac{x^2-16}{4\sqrt{x}-8} = \lim_{x \to 4} \frac{2x}{2/\sqrt{x}} = 8$$

3.3 Applications

3.3.1 Maxima and Minima



local maximum $f'(x_0) = 0$ and $f''(x_0) < 0$

$f'(x_0) = 0$ and $f'(x_0)$ changes its sign at x_0 from $+$ to $-$

local minimum $f'(x_0) = 0$ and $f''(x_0) > 0$

$f'(x_0) = 0$ and $f'(x_0)$ changes its sign at x_0 from $-$ to $+$

Caution The derivative f' of $f(x) = |x|$ at $x_0 = 0$ does not exist, but f has a global minimum at $x_0 = 0$. For $f(x) = x^3$ we have $f'(0) = 0$, but $x_0 = 0$ is neither a minimum nor a maximum. Extrema can lie on the border of the domain of f, where $f'(x) \neq 0$.

3.3.2 Inflection Points

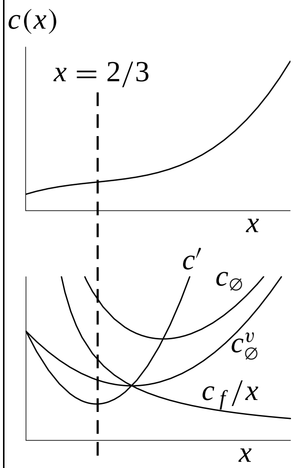
Wendepunkte

inflection point $f''(x_0) = 0$ and $f'''(x_0) \neq 0$

$f''(x_0) = 0$ and $f''(x_0)$ changes its sign at x_0

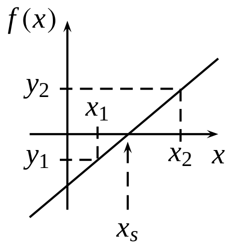
Example (Cost Function)

cost function	$c(x) = x^3 - 2x^2 + 2x + 1$
fixed cost	$c(0) = c_f = 1$
marginal cost	$c'(x) = 3x^2 - 4x + 2$
	$c''(x) = 6x - 4$ *and* $c'''(x) = 6$
average cost	$c_\varnothing(x) = \dfrac{c(x)}{x} = x^2 - 2x + 2 + \dfrac{1}{x}$
variable average cost	$c^v_\varnothing(x) = \dfrac{c(x) - c_f}{x} = x^2 - 2x + 2$
average fixed cost	$c_f/x = 1/x$
inflection point of c	$x = 2/3: \ c''(2/3) = 0, \ c'''(2/3) = 6 > 0$
minimum of c'	$x = 2/3: \ c''(2/3) = 0, \ c'''(2/3) = 6 > 0$
minimum of $c_\varnothing$	$c'_\varnothing(x) = \dfrac{c'(x)x - c(x)}{x^2} = 0$
	$= 2x - 2 - x^{-2} = 0 \implies x \approx 1.297$
fixed cost degession	$(c_f/x)'(x) = -c_f\, x^{-2} < 0$



3.3.3 Zeros of Functions

Nullstellen



☞ *Sec. 1.3.4 (roots of polynomials)*

approximations of $f(x) = 0$

$$x_s = x_1 - y_1 \frac{x_2 - x_1}{y_2 - y_1} \qquad (f(x_1)\, f(x_2) < 0)$$

Newton's approximation method Define a sequence $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$(f'(x_n) \neq 0; \ f(x_n)\, f''(x_n) > 0;$ sign of $f''(x)$ does not switch$)$

3.3.4 Elasticities

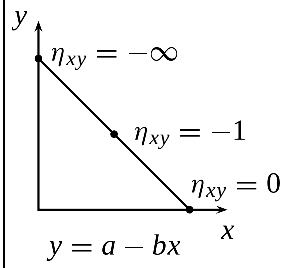
elasticity (in an interval) $\eta_{yx} = \dfrac{\text{relative change of } y \text{ (effect)}}{\text{relative change of } x \text{ (cause)}} = \dfrac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \dfrac{\Delta y}{\Delta x}\dfrac{x}{y}$ — *Bogenelastizität*

elasticity (at a point) $\eta_{yx} = \dfrac{x f'(x)}{f(x)}$ — *Punktelastizität*

elasticity of the inverse Let g be the inverse of $y = f(x)$. Then, $\eta_{yx} = \dfrac{1}{\eta_{xy}}$

Example *The demand function of a good x_1 depends on commodity prices p_1, p_2, and income y.*

demand function	$x_1 = f(p_1, p_2, y)$
direct price elasticity	$\eta_{x_1 p_1} = \dfrac{\Delta x_1}{\Delta p_1} \dfrac{p_1}{x_1}$
cross price elasticity	$\eta_{x_1 p_2} = \dfrac{\Delta x_1}{\Delta p_2} \dfrac{p_2}{x_1}$
income elasticity	$\eta_{x_1 y} = \dfrac{\Delta x_1}{\Delta y} \dfrac{y}{x_1}$
factor elasticities	$x = v_1^a v_2^b \implies \eta_{xv_1} = a, \quad \eta_{xv_2} = b$
elasticity of scale	$x(\lambda) = f(\lambda \bar{v}_1, \lambda \bar{v}_2) \implies \eta_{x\lambda} = \dfrac{\Delta x}{\Delta \lambda} \dfrac{\lambda}{x}$



3.4 Integral Calculus

3.4.1 Indefinite Integrals

unbestimmte Integrale

A function F is an indefinite integral, or *antiderivative*, of f if $F'(x) = f(x)$.

Stammfunktion

$$\int f(x)\mathrm{d}x = F(x) + C \quad \text{when} \quad F'(x) = f(x)$$

The dx term denotes that x is the variable of integration, $f(x)$ is called the integrand, and C is the constant of integration.

$f(x) = F'(x)$	$F(x)$	$f(x) = F'(x)$	$F(x)$
c	cx	$\sin x$	$-\cos x$
x^n	$\dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$	$\sin^2 x$	$\frac{1}{2}(x - \sin x \cos x)$
$1/x$	$\ln\|x\| \quad (x \neq 0)$	$\cos x$	$\sin x$
e^{ax}	$\dfrac{1}{a}\mathrm{e}^{ax}$	$\cos^2 x$	$\frac{1}{2}(x + \sin x \cos x)$
a^x	$\dfrac{a^x}{\ln a} \quad (a > 0, a \neq -1)$	$\tan x$	$-\ln\|\cos x\|$
$\ln x$	$x \ln x - x \quad (x > 0)$	$\tan^2 x$	$\tan x - x$
$\dfrac{1}{x-a}$	$x \ln(x - a)$	$\cot x$	$\ln\|\sin x\|$
$\dfrac{1}{(x-a)(x-b)}$	$\dfrac{1}{a-b} \ln\left\|\dfrac{x-a}{x-b}\right\| \quad (a \neq b)$	$\cot^2 x$	$-\cot x - x$
$\dfrac{1}{(x-a)^2}$	$-\dfrac{1}{x-a}$	$\dfrac{g'(x)}{g(x)}$	$\ln\|g(x)\|$

$$\int a f(x)\,\mathrm{d}x = a \int f(x)\,\mathrm{d}x$$

$$\int [f(x) \pm g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

3.4.2 Definite Integrals

bestimmte Integrale

main theorem (given the lower and upper limits of integration)

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b \quad \text{where} \quad F'(x) = f(x)$$

limits of integration

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x; \qquad \int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

integration by parts

partielle Integration

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = [f(x)g(x)]\Big|_a^b - \int_a^b f(x)g'(x)\,\mathrm{d}x$$

integration by substitution

Substitutionsregel

$$\int_a^b f(g(z))g'(z)\,\mathrm{d}z = \int_{g(a)}^{g(b)} f(x)\,\mathrm{d}x \quad \text{where} \quad x = g(z),\ g'(x) = \frac{\mathrm{d}x}{\mathrm{d}z}$$

3.4.3 Differential Equations

Differential-gleichungen

In differential equations the unknown x is a function (often of time t), but not a number as in ordinary algebraic equations. Moreover, the equation may include one or more derivatives of the unknown function.

notation

$$\dot{x}(t) \equiv \frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

$$\dot{x}(t) = f(t) \iff x(t) = F(t) = \int f(t)\,\mathrm{d}t + C \quad \text{where} \quad F'(t) = f(t)$$

A known value $x(t_0)$ determines the constant of integration C.
separable differential equations

$$\dot{x}(t) = f(t,x) = b(t)g(x) \implies \int \frac{1}{g(x)}\,\mathrm{d}x = \int b(t)\,\mathrm{d}t + C$$

first order linear differential equation

$$\dot{x}(t) + a(t)x(t) = b(t) \iff x(t) = \mathrm{e}^{-\int a(t)\mathrm{d}t}\left[\int \mathrm{e}^{\int a(t)\mathrm{d}t}\, b(t)\,\mathrm{d}t + C\right]$$

If $a(t) = a = \text{const.}$, then $\mathrm{e}^{\int a\,\mathrm{d}t} = \mathrm{e}^{at}$. If furthermore $b(t) = b = \text{const.}$, then $x(t) = C\mathrm{e}^{-at} + b/a$.

second order differential equation $\ddot{x} = f(\dot{x}, x, t)$
second order linear differential equation $\ddot{x} + a(t)\dot{x} + b(t)x = c(t)$
The differential equation is called homogeneous if $c(t) = 0$, and nonhomogeneous otherwise.
General solution of a homogeneous linear differential equation of order two with constant coefficients

$$\ddot{x} + a\dot{x} + bx = 0 \quad (\text{define} \quad D := a^2/4 - b)$$

$D > 0$:	$x = C_1\,\mathrm{e}^{r_1 t} + C_2\,\mathrm{e}^{r_2}\,t$,	where	$r_{1,2} = -a/2 \pm \sqrt{D}$
$D = 0$:	$x = (C_1 + C_2\,t)\,\mathrm{e}^{rt}$,	where	$r = -a/2$
$D < 0$:	$x = C_1\,\mathrm{e}^{\alpha\,t}(\cos\beta t - C_2)$,	where	$\alpha = -a/2,\ \beta = \sqrt{-D}$

4 Calculus of Several Variables

4.1 Functions from $\mathbb{R}^n$ to $\mathbb{R}$

The function $f: \mathcal{X} \to \mathbb{R}$ has a domain $\mathcal{X} \subset \mathbb{R}^n$. That is, each element, or vector, $\mathbf{x} = (x_1, \dots, x_n)$ of $\mathcal{X}$ is assigned to one element of $\mathbb{R}$.

$$y = f(\mathbf{x}) = f(x_1, \dots, x_n)$$

Examples (production functions) *one output x, two (or more) inputs v_1, v_2*

linear	$x = a_1 v_1 + a_2 v_2$
Cobb-Douglas	$x = A\, v_1^a v_2^b$
Leontief	$x = \min\{v_1/a_1,\, v_2/a_2\}$
CES	$x = A(a_1 v_1^{-a} + a_2 v_2^{-a})^{-b/a}$

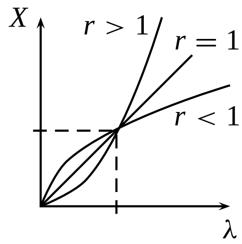
Homogeneity The function f is said to be *homogeneous* of degree r if

Homogenität

$$y = f(x_1, x_2) \implies \lambda^r y = f(\lambda x_1, \lambda x_2) \quad \forall\, \lambda > 0,\ \forall\, x_1, x_2$$

☞ *Euler's theorem*

Examples (returns to scale)



increasing returns to scale $f(\lambda v_1, \lambda v_2) > \lambda f(v_1, v_2)\ \forall \lambda > 1,\ \forall v_1, v_2$

constant returns to scale $f(\lambda v_1, \lambda v_2) = \lambda f(v_1, v_2)\ \forall \lambda > 0,\ \forall v_1, v_2$

decreasing returns to scale $f(\lambda v_1, \lambda v_2) < \lambda f(v_1, v_2)\ \forall \lambda > 1,\ \forall v_1, v_2$

(homogeneous) Cobb-Douglas function

$$x = v_1^a v_2^b \implies \lambda^r x = (\lambda\, v_1)^a (\lambda\, v_2)^b = \lambda^{a+b} v_1^a v_2^b \implies r = a + b$$

economies of scale: $r > 1$
production level λ: $x = \lambda^r \bar{x} = f(\lambda \bar{v}_1, \lambda \bar{v}_2),\ \bar{x} = f(\bar{v}_1, \bar{v}_2)$
elasticity of scale: $\eta_{x\lambda} = r$

4.2 Partial Derivatives

Differentiability The function f is differentiable at point (x_1^*, x_2^*) with respect to x_1 if the following limit exists.

Differenzierbarkeit

$$\lim_{h \to 0} \frac{f(x_1^* + h, x_2^*) - f(x_1^*, x_2^*)}{h} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}$$

notation:
$\mathrm{d} \to \partial$

The function $\partial f/\partial x_j$ is the *partial derivative* of f with respect to x_j $(j = 1, 2)$. The term $\frac{\partial f}{\partial x_j}(x_1^*, x_2^*)$ denotes the functional value of $\partial f/\partial x_j$ at the point (x_1^*, x_2^*).

partielle Ableitung

gradient ∇f of f at $\mathbf{x}$ $\quad \nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right)^{\mathsf{T}}$

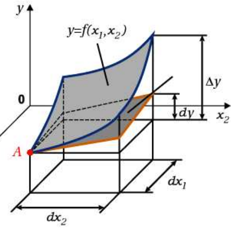
Gradient

The gradient ∇f points in the direction of maximal increase of f.
A function $f: \mathbb{R}^n \to \mathbb{R}$ is *continuously differentiable* if all partial derivatives $(\partial f/\partial x_j)(\mathbf{x})$ exist and are continuous in $\mathbf{x}$.

stetig differenzierbar

total differential $\mathrm{d}y = \frac{\partial f(x_1, x_2)}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\,\mathrm{d}x_2$

totales Differential



chain rule $F(x_1, x_2) = f\big(g_1(x_1, x_2), g_2(x_1, x_2)\big)$

$$\Longrightarrow \quad \frac{\partial F}{\partial x_j} = \frac{\partial f}{\partial g_1}\frac{\partial g_1}{\partial x_j} + \frac{\partial f}{\partial g_2}\frac{\partial g_2}{\partial x_j} \quad (j = 1, 2)$$

Example (Wicksell-Johnson theorem) *Let* $x = f(v_1, v_2)$ *be a production function and define the scale function* $x(\lambda) = f(\lambda\bar{v}_1, \lambda\bar{v}_2)$ *with* $v_1 = \lambda\bar{v}_1$ *and* $v_2 = \lambda\bar{v}_2$. *Then,*

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\lambda}{x} = \left[\frac{\partial f}{\partial v_1}\frac{\mathrm{d}v_1}{\mathrm{d}\lambda} + \frac{\partial f}{\partial v_2}\frac{\mathrm{d}v_2}{\mathrm{d}\lambda}\right]\frac{\lambda}{x} = \left[\frac{\partial f}{\partial v_1}\bar{v}_1 + \frac{\partial f}{\partial v_2}\bar{v}_2\right]\frac{\lambda}{x} = \frac{\partial f}{\partial v_1}\frac{v_1}{x} + \frac{\partial f}{\partial v_2}\frac{v_2}{x}$$

The Wicksell-Johnson theorem states that the scale elasticity equals the sum of all factor elasticities, i.e., $\eta_{x\lambda} = \eta_{xv_1} + \eta_{xv_2}$.

Euler's theorem Let f be homogeneous of degree r, then

Euler Theorem

$$rf(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_1}\,x_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\,x_2.$$

Homogeneity of derivatives Let f be homogeneous of degree r, then the partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ are homogeneous of degree $r - 1$.

Example (production function) $x = f(v_1, v_2) = v_1^a v_2^b$

$$\mathrm{d}x = \frac{\partial f(v_1, v_2)}{\partial v_1}\,\mathrm{d}v_1 + \frac{\partial f(v_1, v_2)}{\partial v_2}\,\mathrm{d}v_2 = av_1^{a-1}v_2^b\,\mathrm{d}v_1 + bv_1^a v_2^{b-1}\,\mathrm{d}v_2$$

$$\lambda^r x = (\lambda\, v_1)^a(\lambda\, v_2)^b = \lambda^{a+b}v_1^a v_2^b \implies r = a + b$$

$$rx = av_1^{a-1}v_2^b\,v_1 + bv_1^a v_2^{b-1}\,v_2 = (a+b)\,x$$

When factors are paid in accordance with their monetary marginal productivity

$\partial f/\partial v_i$
= marginal productivity
= Grenzproduktivität

$$q_1 = p\frac{\partial f(v_1, v_2)}{\partial v_1} \quad \text{and} \quad q_2 = p\frac{\partial f(v_1, v_2)}{\partial v_2},$$

then $rpx = q_1v_1 + q_2v_2$. *For a linear homogeneous production function with* $r = 1$ *this is the* adding up theorem *(revenue = cost).*

Adding-Up Theorem

implicit function theorem or implicit differentiation

$$0 = f(x_1, x_2) \implies \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}}$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} \neq 0$$

second order partial derivatives of f (generalization of $f''(x)$)

Hessean matrix $$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & \cdots & f''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{n1} & f''_{n2} & \cdots & f''_{nn} \end{pmatrix}$$

notation:

$$f''_{ij} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Young's (or Schwarz's) theorem (symmetric cross (or mixed) partial derivatives)

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2} = \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \, \partial x_1}$$

Example (utility function)

$$U = u(x_1, x_2) = x_1^a x_2^b$$
$$\Longrightarrow \frac{\partial u(x_1, x_2)}{\partial x_1} = a x_1^{a-1} x_2^b \quad \textit{and} \quad \frac{\partial u(x_1, x_2)}{\partial x_2} = b x_1^a x_2^{b-1}$$
$$\Longrightarrow \frac{\partial^2 u(x_1, x_2)}{\partial x_1 \, \partial x_2} = \frac{\partial^2 u(x_1, x_2)}{\partial x_2 \, \partial x_1} = ab x_1^{a-1} x_2^{b-1}$$

marginal utility = Grenznutzen

Convexity Let f be a function of two variables with continuous partial derivatives of the first and the second order. Then

Konvexität

(a) f is *convex* $\iff f''_{11} \geq 0,\ f''_{22} \geq 0,$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

(b) f is *concave* $\iff f''_{11} \leq 0,\ f''_{22} \leq 0,$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

☞ *Sec. 5.2.3*

Replace all $\geq$ and $\leq$ by $>$ and $<$ for strict convexity and strict concavity.

4.3 Unconstrained Optimization

The least upper bound $\overline{B}$ such that $\overline{B} \geq f(x)$ for all $x \in \mathcal{X}$ is referred to as *supremum* of f on $\mathcal{X}$ (with $\mathcal{X}$ being the domain of f); the supremum is denoted by $\sup\{f(x) | \, x \in \mathcal{X}\}$. Similarly, the greatest lower bound $\underline{B}$ such that $\underline{B} \leq f(x)$ for all $x \in \mathcal{X}$ is called the *infimum* of f on $\mathcal{X}$; the infimum is denoted by $\inf\{f(x) | \, x \in \mathcal{X}\}$. There is no need that f attaines the supremum or the infimum at any point. But if there is some point $\hat{x}$ such that $f(\hat{x}) = \sup\{f(x) | \, x \in \mathcal{X}\}$ is finite, then $\hat{x}$ is called a *maximum point* and f attains its *maximum value* $\max\{f(x) | \, x \in \mathcal{X}\}$ at $\hat{x}$. The same holds true for the minimum value at a minimum point.

Supremum

Infimum

Maximum

Necessary first-order conditions Let f be differentiable and $\mathbf{x}_0$ be an interior point of the domain of f. A necessary condition for $\mathbf{x}_0$ to be a maximum or minimum point of f is that $\mathbf{x}_0$ is a *stationary, or critical, point* for f; that is

☞ *Sec. 3.3.1*

stationärer Punkt

$$\frac{\partial f(\mathbf{x}_0)}{\partial x_1} = 0, \ldots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n} = 0$$

Second-order conditions Let f have continuous partial derivatives and let $\mathbf{x}_0$ be an interior point of the (convex) domain.

(a) If f is *convex*, then $\mathbf{x}_0$ is a (global) minimum point of f if and only if $\mathbf{x}^0$ is a stationary point of f.

(b) If f is *concave*, then $\mathbf{x}_0$ is a (global) maximum point of f if and only if $\mathbf{x}^0$ is a stationary point of f.

☞ *Sec. 4.2* *and* ☞ *Sec. 5.2.3*

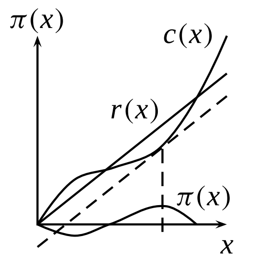
Example (profit maximization)

(a) revenue $r(x) = px$, cost function $c(x)$

$$\pi(x) = r(x) - c(x) \to \max \implies \pi'(x) = p - c'(x) = 0 \iff p = c'(x)$$

(b) revenue $r(v_1, v_2) = pf(v_1, v_2)$, factor costs $c = q_1v_1 + q_2v_2$

$$\pi(v_1, v_2) = pf(v_1, v_2) - q_1v_1 - q_2v_2 \to \max$$
$$\implies \frac{\partial\pi(v_1, v_2)}{\partial v_1} = 0 \iff p\frac{\partial f(v_1, v_2)}{\partial v_1} = q_1$$
$$\text{and} \quad \frac{\partial\pi(v_1, v_2)}{\partial v_2} = 0 \iff p\frac{\partial f(v_1, v_2)}{\partial v_2} = q_2$$



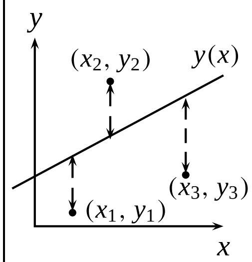
Example (least square analysis)

Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, what line $y = ax + b$ fits best to these data? Idea: minimize the sum of all absolute distances between observed values y_i and expected values $y(x_i) = ax_i + b$ with respect to the parameters a and b. This is equivalent to

$$\min_{a,b} S(a, b) \quad \text{with} \quad S(a, b) = \textstyle\sum_{i=1}^{n}(ax_i + b - y_i)^2$$

Solve the following system for a and b by using the observed data.

$$\frac{\partial S}{\partial a} = \textstyle\sum_i 2(ax_i + b - y_i)\,x_i = 0 \iff a\sum_i x_i^2 + b\sum_i x_i = \sum_i y_ix_i$$
$$\frac{\partial S}{\partial b} = \textstyle\sum_i 2(ax_i + b - y_i) = 0 \iff a\sum_i x_i + b\,n = \sum_i y_i$$



Envelope theorem Let $f(\mathbf{x}; a)$ be a function of $\mathbf{x} \in \mathbb{R}^n$ and the parameter a. For each choice of a solve the unconstrained maximum problem $\max_{\mathbf{x}} f(\mathbf{x}; a)$. The solution $\hat{\mathbf{x}}(a)$ is a function of a; if it is continuously differentiable in a, then

Umhüllendensatz

$$\frac{\mathrm{d}}{\mathrm{d}a} f(\hat{\mathbf{x}}(a); a) = \frac{\partial}{\partial a} f(\hat{\mathbf{x}}(a); a).$$

Rule of thumb "In analyzing variations of a we can treat $\hat{\mathbf{x}}(a)$ as if it is constant."

application see Sec. 4.4.5

4.4 Constrained Optimization

4.4.1 General Problems

Consider the problem of maximizing (or minimizing) an objective, or criterion, function ① when the variables x_1 and x_2 are restricted to satisfy several constraints. The restrictions can be equations, such as ②, or inequalities, such as ③ or ④ (non-negativity of variables).

Zielfunktion

Nebenbedingungen

①	$\max \ f(x_1, x_2)$	$\min \ f(x_1, x_2)$
②	s.t. $g_1(x_1, x_2) = 0$	s.t. $g_1(x_1, x_2) = 0$
③	$g_2(x_1, x_2) \geq 0$	$g_2(x_1, x_2) \leq 0$
④	$x_1, x_2 \geq 0$	$x_1, x_2 \geq 0$

4.4.2 Substitution

$$\left.\begin{array}{ll} \max & f(x_1, x_2) \\ \text{s.t.} & x_1 - g(x_2) = 0 \end{array}\right\} \Longrightarrow \max h(x_2) \quad \text{with} \quad h(x_2) \equiv f(g(x_2), x_2)$$

A necessary condition for a maximum of h at $\hat{x}_2$ is $h'(\hat{x}_2) = 0$, where

$$h'(x_2) = \frac{\partial f(g(x_2), x_2)}{\partial x_1} \frac{\mathrm{d}g(x_2)}{\mathrm{d}x_2} + \frac{\partial f(g(x_2), x_2)}{\partial x_2}$$

Example (utility maximization)

$$\left.\begin{array}{ll} \max & x_1^a x_2^b \\ s.t. & y = p_1 x_1 + p_2 x_2 \end{array}\right\} \Longrightarrow \max \left(\frac{y}{p_1} - \frac{p_2}{p_1} x_2\right)^a x_2^b$$

$x_1 = \frac{y}{p_1} - \frac{p_2}{p_1} x_2$

$\hat{x}_1 = \frac{y}{p_1} - \frac{p_2}{p_1} \hat{x}_2$

$$\begin{aligned} & a\,\hat{x}_1{}^{a-1} \hat{x}_2{}^b \left(-\frac{p_2}{p_1}\right) + \hat{x}_1{}^a\, b\, \hat{x}_2{}^{b-1} = 0 \\ \Longleftrightarrow \quad & a\,\hat{x}_2 \left(-\frac{p_2}{p_1}\right) + \left(\frac{y}{p_1} - \frac{p_2}{p_1}\hat{x}_2\right) b = 0 \\ \Longleftrightarrow \quad & (y - p_2\hat{x}_2) b = a\hat{x}_2 p_2 \\ \Longleftrightarrow \quad & yb = (a+b) p_2 \hat{x}_2 \\ \Longleftrightarrow \quad & \hat{x}_2 = \frac{y}{p_2}\frac{b}{a+b} \Longrightarrow \cdots \Longrightarrow \hat{x}_1 = \frac{y}{p_1}\frac{a}{a+b} \end{aligned}$$

in general:
$\hat{x}_1 = x_1(p_1, p_2, y)$
$\hat{x}_2 = x_2(p_1, p_2, y)$

4.4.3 Lagrangean Method

Lagrange Methode

Both problems in Sec. 4.4.1 have the same Lagrange function L with Lagrangean multipliers λ_1, λ_2.

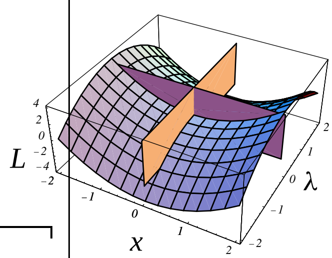
$$L(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) + \lambda_1 g_1(x_1, x_2) + \lambda_2 g_2(x_1, x_2)$$

In the maximum problem, L is to be maximized with respect to x_1 and x_2 and, at the same time, to be minimized with respect to λ_1 and λ_2. The reverse holds true for a minimum problem. $\Longrightarrow$ Apply first-order conditions of Sec. 4.3!

A vector $(\hat{x}_1, \hat{x}_2, \hat{\lambda}_1, \hat{\lambda}_2)$ is a *saddle point* of L if for every $(x_1, x_2, \lambda_1, \lambda_2)$

Sattelpunkt

$$\begin{aligned} & L(x_1, x_2, \hat{\lambda}_1, \hat{\lambda}_2) \le L(\hat{x}_1, \hat{x}_2, \hat{\lambda}_1, \hat{\lambda}_2) \le L(\hat{x}_1, \hat{x}_2, \lambda_1, \lambda_2) \\ \text{or} \quad & L(x_1, x_2, \hat{\lambda}_1, \hat{\lambda}_2) \ge L(\hat{x}_1, \hat{x}_2, \hat{\lambda}_1, \hat{\lambda}_2) \ge L(\hat{x}_1, \hat{x}_2, \lambda_1, \lambda_2) \end{aligned}$$



optimal solution $\quad L(\hat{x}_1, \hat{x}_2, \hat{\lambda}_1, \hat{\lambda}_2) = f(\hat{x}_1, \hat{x}_2)$

Example *The same example as above but with non-negative values of x_1, x_2.*

$$\begin{aligned} \max\ & x_1^a x_2^b \\ s.t.\ & y = p_1 x_1 + p_2 x_2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Lagrange function L with Lagrangean multiplier λ

$$L(x_1, x_2, \lambda) = x_1^a x_2^b + \lambda(y - p_1 x_1 - p_2 x_2)$$

first order conditions

$$\frac{\partial L}{\partial x_1} = a\hat{x}_1{}^{a-1}\hat{x}_2{}^b - \hat{\lambda} p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = \hat{x}_1{}^a b\hat{x}_2{}^{b-1} - \hat{\lambda} p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = y - p_1\hat{x}_1 - p_2\hat{x}_2 = 0$$

3 equations, 3 variables ($\hat{x}_1, \hat{x}_2, \hat{\lambda}$)

eliminate $\hat{\lambda}$

$$\frac{p_1}{p_2} = \frac{a\hat{x}_2}{b\hat{x}_1} \iff \hat{x}_1 = \frac{ap_2}{bp_1}\hat{x}_2$$
$$y = p_1\hat{x}_1 + p_2\hat{x}_2$$

2 equations 2 variables ($\hat{x}_1, \hat{x}_2$)

eliminate $\hat{x}_1$

$$y = p_1\left(\frac{ap_2}{bp_1}\hat{x}_2\right) + p_2\hat{x}_2 = \left(\frac{a}{b} + 1\right)p_2\hat{x}_2 \iff \hat{x}_2 = \frac{y}{p_2}\frac{b}{a+b}$$

1 equation 1 variable ($\hat{x}_2$) same result as before

4.4.4 Karush-Kuhn-Tucker (KKT) Conditions

The following procedure ensures nonnegative Lagrangean multipliers λ_1 and λ_2!

(1) minimum problem $\min_{\mathbf{x}} f(\mathbf{x})$ s.t. $g_1(\mathbf{x}) \le 0$, $g_2(\mathbf{x}) \le 0$, $\mathbf{x} \geqq \mathbf{0}$

Lagrange function $L(\mathbf{x}, \lambda_1, \lambda_2) = f(\mathbf{x}) + \lambda_1 g_1(\mathbf{x}) + \lambda_2 g_2(\mathbf{x})$

KKT conditions

$$\hat{x}_j \ge 0, \quad \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial x_j} \ge 0, \quad \hat{x}_j \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial x_j} = 0 \qquad j = 1, \dots, m$$
$$\hat{\lambda}_i \ge 0, \quad \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial \lambda_i} = g_i(\hat{\mathbf{x}}) \le 0, \quad \hat{\lambda}_i \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial \lambda_i} = 0 \qquad i = 1, 2$$

(2) maximum problem $\max_{\mathbf{x}} f(\mathbf{x})$ s.t. $g_1(\mathbf{x}) \ge 0$, $g_2(\mathbf{x}) \ge 0$, $\mathbf{x} \geqq \mathbf{0}$

Lagrange function $L(\mathbf{x}, \lambda_1, \lambda_2) = f(\mathbf{x}) + \lambda_1 g_1(\mathbf{x}) + \lambda_2 g_2(\mathbf{x})$

KKT conditions

$$\hat{x}_j \ge 0, \quad \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial x_j} \le 0, \quad \hat{x}_j \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial x_j} = 0 \qquad j = 1, \dots, m$$
$$\hat{\lambda}_i \ge 0, \quad \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial \lambda_i} = g_i(\hat{\mathbf{x}}) \ge 0, \quad \hat{\lambda}_i \frac{\partial L(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)}{\partial \lambda_i} = 0 \qquad i = 1, 2$$

Theorem Suppose that f, g_1, and g_2 are all differentiable and that the Lagrange function L is convex (concave) in $\mathbf{x}$. Assume furthermore that a vector $(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)$ exists which satisfies the above KKT conditions, then $\hat{\mathbf{x}}$ solves the minimum (maximum) problem. Moreover, $(\hat{\mathbf{x}}, \hat{\lambda}_1, \hat{\lambda}_2)$ is a saddle point of L.

☞ Bazaraa, Sherali, Shetty (1993) for more details.

4.4.5 Economic Applications

(1) Expenditure minimization Let $\hat{x}_1 = x_1(p_1, p_2, U)$ and $\hat{x}_2 = x_2(p_1, p_2, U)$ solve the expenditure minimization problem

$$\begin{aligned} &\min\{\, p_1x_1 + p_2x_2 \mid U \leq u(x_1, x_2),\ x_1 \geq 0,\ x_2 \geq 0\,\} \\ \Longrightarrow\quad &L(x_1, x_2, \lambda;\, p_1, p_2, U) = p_1x_1 + p_2x_2 + \lambda(U - u(x_1, x_2)) \end{aligned}$$

Then, $e(p_1, p_2, U) \coloneqq p_1x_1(p_1, p_2, U) + p_2x_2(p_1, p_2, U)$, and, by the envelope theorem,

$$\begin{aligned} L(\hat{x}_1, \hat{x}_2, \hat{\lambda};\, p_1, p_2, U) &= p_1\hat{x}_1 + p_2\hat{x}_2 = e(p_1, p_2, U) \\ &\Longrightarrow\ \frac{\partial e}{\partial U} = \frac{\partial L}{\partial U} = \hat{\lambda} \\ \text{Shephard's lemma} \qquad &\Longrightarrow\ \frac{\partial e}{\partial p_1} = \frac{\partial L}{\partial p_1} = \hat{x}_1 \\ \text{Shephard's lemma} \qquad &\Longrightarrow\ \frac{\partial e}{\partial p_2} = \frac{\partial L}{\partial p_2} = \hat{x}_2 \end{aligned}$$

Shephards Lemma

(2) Utility maximization Let $\hat{x}_1 = x_1(p_1, p_2, y)$ and $\hat{x}_2 = x_2(p_1, p_2, y)$ solve the expenditure minimization problem

$$\begin{aligned} &\min\{\, u(x_1, x_2) \mid y \geq p_1x_1 + p_2x_2,\ x_1 \geq 0,\ x_2 \geq 0\,\} \\ \Longrightarrow\quad &L(x_1, x_2, \lambda;\, p_1, p_2, y) = u(x_1, x_2) + \lambda(y - p_1x_1 - p_2x_2) \end{aligned}$$

Then, $v(p_1, p_2, y) \coloneqq u\big(x_1(p_1, p_2, y), x_2(p_1, p_2, y)\big)$, and, by the envelope theorem,

$$\begin{aligned} L(\hat{x}_1, \hat{x}_2, \hat{\lambda};\, p_1, p_2, y) &= u(\hat{x}_1, \hat{x}_2) = v(p_1, p_2, y) \\ &\Longrightarrow\ \frac{\partial v}{\partial y} = \frac{\partial L}{\partial y} = \hat{\lambda} \\ &\Longrightarrow\ \frac{\partial v}{\partial p_1} = \frac{\partial L}{\partial p_1} = -\hat{\lambda}\,\hat{x}_1 \ \Longrightarrow\ \hat{x}_1 = -\frac{\partial v/\partial p_1}{\partial v/\partial y} \\ &\Longrightarrow\ \frac{\partial v}{\partial p_2} = \frac{\partial L}{\partial p_2} = -\hat{\lambda}\,\hat{x}_2 \ \Longrightarrow\ \hat{x}_1 = -\frac{\partial v/\partial p_2}{\partial v/\partial y} \end{aligned}$$

Grenznutzen des Geldes

5 Linear Algebra

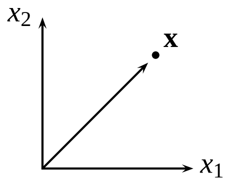
5.1 Vectors and Matrices

scalar	$a \in \mathbb{R}$	*Skalar*
(row) vector	$(x_1, \dots, x_n) \in \mathbb{R}^n$	*(Zeilen-)Vektor*
(column) vector	$\mathbf{x} = (x_1, \dots, x_n)^\mathsf{T} \in \mathbb{R}^n$	*(Spalten-)Vektor*
matrix	$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^3$	*Matrix*
$m \times n$-matrix	$\mathbf{A} = (a_{ij})_{mn}$; it has m rows and n columns matrix of order $m \times n$	*Dimension*
square matrix	$\mathbf{A} = (a_{ij})_{nn}$; it has n rows and n columns	

Transposition of a row vector generates a column vector, and vice versa.

$$(\mathbf{x}^\mathsf{T})^\mathsf{T} = \mathbf{x}$$

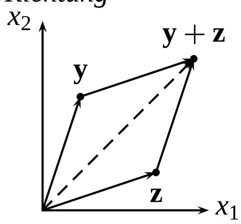
A *point* $\mathbf{x} = (x_1, \dots, x_n)^\mathsf{T}$ may be seen as a vector that points from the origin $\mathbf{0} = (0, \dots, 0)^\mathsf{T}$ to the point $\mathbf{x}$. *Punkt*



Multiplication by a scalar a

$$a\,\mathbf{x} = (ax_1, \dots, ax_n)^\mathsf{T}$$

yields another vector that points into the same *direction* if $a > 0$, or into the opposite direction if $a < 0$. *Richtung*



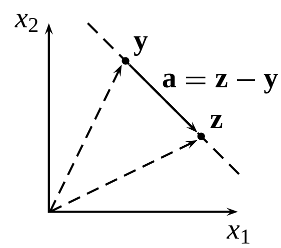
The sum of two vectors

$$\mathbf{y} + \mathbf{z} = (y_1 + z_1, \dots, y_n + z_n)^\mathsf{T}$$

can be represented by a parallelogram.

Any *line* through $\mathbf{y}$ can be described by $\mathbf{y}$ and a vector $\mathbf{a}$ indicating the direction of the line. *Gerade*

$$\mathbf{x} = \mathbf{y} + t\,\mathbf{a} \quad \forall\, t \in \mathbb{R}$$



(If $t \geqq 0$, we speak of a *ray* starting at $\mathbf{y}$ into the direction of $\mathbf{a}$.)
The line through $\mathbf{y}$ and $\mathbf{z}$ is given by

$$\mathbf{x} = (1-t)\mathbf{y} + t\,\mathbf{z} \quad \forall\, t \in \mathbb{R}$$

The *line segment* joining $\mathbf{y}$ and $\mathbf{z}$ follows from *Strecke*

$$\overline{\mathbf{yz}} = [\mathbf{y}, \mathbf{z}] = \{\, \mathbf{x} \mid \mathbf{x} = (1-t)\mathbf{y} + t\,\mathbf{z},\ 0 \le t \le 1 \,\}\,.$$

Any *plane* can be described by three vectors which must have different directions: *Ebene*

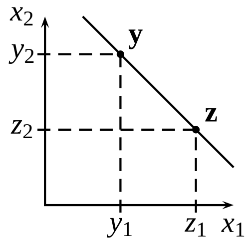
$$\mathbf{x} = \mathbf{y} + a_1\,\mathbf{z}_1 + a_2\,\mathbf{z}_2 \quad \forall\, a_1, a_2 \in \mathbb{R}$$

point point formula $\frac{x_2 - y_2}{x_1 - y_1} = \frac{z_2 - y_2}{z_1 - y_1}$

point slope formula $x_2 - y_2 = a(x_1 - y_1); \quad x_2 = m(x_1 - a)$

main form $x_2 = mx_1 + b$

axis intercepts $\frac{x_1}{a} + \frac{x_2}{b} = 1$



scalar product $\mathbf{y}^\mathsf{T}\mathbf{x} = \sum_{j=1}^{n} y_j x_j = y_1x_1 + y_2x_2 + \cdots + y_nx_n \qquad (\mathbf{y}, \mathbf{x} \in \mathbb{R}^n)$

Skalarprodukt

$$\mathbf{y}^\mathsf{T}\mathbf{x} = \mathbf{x}^\mathsf{T}\mathbf{y}, \quad \mathbf{x}^\mathsf{T}(\mathbf{y}+\mathbf{z}) = \mathbf{x}^\mathsf{T}\mathbf{y} + \mathbf{x}^\mathsf{T}\mathbf{z}, \quad a(\mathbf{x}^\mathsf{T}\mathbf{y}) = (a\mathbf{x})^\mathsf{T}\mathbf{y} = (a\mathbf{y})^\mathsf{T}\mathbf{x}$$

orthogonal, or vertical, vectors: $\mathbf{x}^\mathsf{T}\mathbf{y} = 0 \iff \mathbf{x} \perp \mathbf{y}$ or $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$

Euclidean norm	$\lvert\mathbf{x}\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$	= length of $\mathbf{x} \in \mathbb{R}^n$	*Länge*
unit vector	$\lvert\mathbf{x}\rvert = 1$	e.g., $\mathbf{e}_1 = (1, 0, \ldots, 0)^\mathsf{T}$	*Einheitsvektor*
basic rules	$\lvert\mathbf{x}\rvert = 0 \iff \mathbf{x} = \mathbf{0}$		
	$\lvert\mathbf{x}+\mathbf{y}\rvert \le \lvert\mathbf{x}\rvert + \lvert\mathbf{y}\rvert$	$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	
	$\lvert\lambda\mathbf{x}\rvert = \lvert\lambda\rvert\,\lvert\mathbf{x}\rvert$	$\forall\, \lambda \in \mathbb{R},\ \forall\, \mathbf{x} \in \mathbb{R}^n$	$\lvert\mathbf{x}\rvert = \lvert-\mathbf{x}\rvert$
Schwarz's inequality	$\lvert\mathbf{y}^\mathsf{T}\mathbf{x}\rvert \le \lvert\mathbf{y}\rvert\,\lvert\mathbf{x}\rvert$	$\forall\, \mathbf{y},\ \mathbf{x} \in \mathbb{R}^n$	

notational convenience for inequalities

$$\begin{aligned} \mathbf{x} > \mathbf{y} &:\iff x_j > y_j \qquad j = 1, \ldots, n; \\ \mathbf{x} \geqq \mathbf{y} &:\iff x_j \ge y_j \qquad j = 1, \ldots, n; \\ \mathbf{x} \ge \mathbf{y} &:\iff [\mathbf{x} \geqq \mathbf{y} \quad \text{and} \quad \mathbf{x} \ne \mathbf{y}]. \end{aligned}$$

5.2 Matrix Operations

5.2.1 Rules of Addition and Multiplication

Two $m \times n$-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are added by adding the corresponding elements:

$$\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$$

commutative law $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

associative law $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$\mathbf{A} = \mathbf{B} \implies \mathbf{A} \pm \mathbf{C} = \mathbf{B} \pm \mathbf{C}$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are* $m \times n$*-matrices*

A matrix $\mathbf{A} = (a_{ij})$ is multiplied by a scalar c by multiplying each element of the matrix by c.

$$c\mathbf{A} = c \cdot (a_{ij}) = (c\, a_{ij})$$

commutative law $c\,\mathbf{A} = \mathbf{A}\,c$

associative law $(c \cdot d)\mathbf{A} = c(d\,\mathbf{A})$

distributive law $c(\mathbf{A}+\mathbf{B}) = c\,\mathbf{A} + c\,\mathbf{B}$ and $(c+d)\mathbf{A} = c\,\mathbf{A} + d\,\mathbf{A}$

The product of an $m \times n$-matrix $\mathbf{A} = (a_{ij})$ and an $n \times r$-matrix $\mathbf{B} = (b_{jk})$ is defined by

$$\mathbf{AB} = \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right)$$

$$\begin{array}{cc} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \end{array}$$

no commutative law in general $\mathbf{AB} \neq \mathbf{BA}$

associative law $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$

distributive law $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ and $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

$\mathbf{A} = \mathbf{B} \implies \mathbf{AC} = \mathbf{BC}$ and $\mathbf{DA} = \mathbf{DB}$

square matrix $\mathbf{A}^2 = \mathbf{AA}, \quad \mathbf{A}^m\mathbf{A}^s = \mathbf{A}^{m+s}, \quad (\mathbf{A}^m)^s = \mathbf{A}^{ms}$

5.2.2 Rules of Transposition

transposition $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$

general rules

$$\begin{aligned} (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} &= \mathbf{A} \\ (\mathbf{A}+\mathbf{B})^{\mathsf{T}} &= \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} \\ (a\,\mathbf{A})^{\mathsf{T}} &= a\,\mathbf{A}^{\mathsf{T}} \\ (\mathbf{AB})^{\mathsf{T}} &= \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \end{aligned}$$

symmetric matrix $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$

5.2.3 Determinants and Matrix Inversion

Determinants The determinant $|\mathbf{A}|$ of an $n \times n$-matrix $\mathbf{A}$ is computed by recursive operations. Let $\mathbf{A}_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from $\mathbf{A}$. Then,

Determinante

application: see Cramer's rule

$$\begin{aligned} |\mathbf{A}| &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| \qquad \text{computation with respect to row } i \\ &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| \qquad \text{computation with respect to column } j \end{aligned}$$

determinants of a 2×2 and 3×3-matrix (computed with respect to the first row)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

general rules $|\mathbf{A}| = |\mathbf{A}^\mathsf{T}|$

$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$

$|a\mathbf{A}| = a^n|\mathbf{A}|$ ($n \times n$-matrix!)

in general $|\mathbf{AB}| \neq |\mathbf{A}| + |\mathbf{B}|$

nonsingular matrix $|\mathbf{A}| \neq 0$ *nichtsinguläre Matrix*

Definiteness A symmetric $n \times n$ matrix $\mathbf{A}$ is *Definitheit*

positive definite if $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} > 0$ $\forall\, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$

negative definite if $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} < 0$ $\forall\, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$

Replace $>$ and $<$ by $\geq$ and $\leq$ for a semidefinite matrix $\mathbf{A}$.

leading principal minors of order k ($k = 1, \ldots, n$) of an $n \times n$ matrix $\mathbf{A}$

$$|\mathbf{A}_1| = |a_{11}|, \quad |\mathbf{A}_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad |\mathbf{A}_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots$$

Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Then,

(a) $\mathbf{A}$ is positive definite if and only if all its n leading principal minors are positive.

(b) $\mathbf{A}$ is negative definite if and only if the n leading principal minors satisfy $(-1)^k\,|\mathbf{A}_k| > 0$, i.e., $|\mathbf{A}_1| < 0$, $|\mathbf{A}_2| > 0$, $|\mathbf{A}_3| < 0$, etc.

Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be twice continuouly differentiable. Then,

(a) f is a convex (concave) function if and only if the Hessean $\mathbf{H}_f(\mathbf{x})$ is positive (negative) semidefinite for all $\mathbf{x} \in \mathbb{R}^n$.

(b) f is a strictly convex (strictly concave) function if and only if the Hessean $\mathbf{H}_f(\mathbf{x})$ is positive (negative) definite for all $\mathbf{x} \in \mathbb{R}^n$.

Inverse The inverse $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$ is a matrix which satisfies *Inverse*

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{E},$$

where $\mathbf{E}$ is the *identity matrix* having the elements $e_{jj} = 1$ and $e_{ij} = 0$ for $i \neq j$. *Einheitsmatrix* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathbf{AE} = \mathbf{EA} = \mathbf{A}$$

$$|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}$$

An $n \times n$ matrix $\mathbf{A}$ is invertible $\iff$ $\mathbf{A}$ is nonsingular.

5.3 Systems of Linear Equations

5.3.1 Gaussian Elimination

linear equations	$\mathbf{Ax} = \mathbf{b}$
invertible	If $\mathbf{A}$ is invertible, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
consistent	If $\mathbf{Ax} = \mathbf{b}$ has at least one solution, it is said to be consistent. (Solutions are not necessarily unique!)
inconsistent	If $\mathbf{Ax} = \mathbf{b}$ has no solution, it is said to be inconsistent.

Linear Independence The m vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ are *linearly independent* if $\lambda_1\mathbf{a}_1 + \cdots + \lambda_m\mathbf{a}_m = \mathbf{0}$ is only satisfied when $\lambda_1 = \cdots = \lambda_m = 0$. *lineare Unabhängigkeit*
There are at the most n linearly independent vectors in $\mathbb{R}^n$.
Rank The *rank* $r(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximum number of linearly independent column vectors of $\mathbf{A}$. *Rang*
Consistency of the system $\mathbf{Ax} = \mathbf{b}$ with $(\mathbf{A}|\mathbf{b})$ denoting the augmented coefficient matrix: *Lösbarkeit*

$$
\begin{aligned}
r(\mathbf{A}) &= r(\mathbf{A}|\mathbf{b}) && \Longleftrightarrow \mathbf{Ax} = \mathbf{b} \text{ is consistent} \\
r(\mathbf{A}) &= r(\mathbf{A}|\mathbf{b}) = r = n && \Longleftrightarrow \mathbf{Ax} = \mathbf{b} \text{ has one and only one solution} \\
r(\mathbf{A}) &= r(\mathbf{A}|\mathbf{b}) = r < n && \Longleftrightarrow \mathbf{Ax} = \mathbf{b} \text{ has more than one solution}
\end{aligned}
$$

In the last case $n - r$ variables can be chosen freely ($n - k$ *degrees of freedom*). *Freiheitsgrade*
Gaussian algorithm (suppose here $m \geq n$)
step 1: determine the system of linear equation $\mathbf{Ax} = \mathbf{b}$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \\
a_{m1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

step 2: expand the coefficient matrix (augmented matrix $(\mathbf{A}|\mathbf{b})$)

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
 & \vdots & & & \\
a_{m1} & a_{n2} & \cdots & a_{mn} & b_m
\end{pmatrix}
$$

step 3: use row operations to derive an *upper triangular matrix*

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
0 & a_{22}^* & a_{23}^* & \cdots & a_{2n}^* & b_2^* \\
0 & 0 & a_{33}^* & \cdots & a_{3n}^* & b_3^* \\
 & \vdots & & & & \\
0 & 0 & 0 & \cdots & a_{nn}^* & b_n^* \\
0 & 0 & 0 & \cdots & 0 & 0 \\
 & \vdots & & & & \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

Row operations (set of solutions does not change):

Zeilenoperationen

(a) Multiply each element of a row by a scalar c ($c \neq 0$)

(b) Add the elements of a row (after multiplication by a scalar) to another row.

(c) Swap two rows.

Example

$$\begin{aligned} x_1 + 5x_2 + 2x_3 + 3x_4 &= 4 \\ 4x_1 + 18x_2 + 2x_3 + 8x_4 &= 12 \\ 3x_1 + 11x_2 - 6x_3 + x_4 &= 4 \\ 2x_2 + 6x_3 + 4x_4 &= 4 \end{aligned}$$

$$\begin{pmatrix} \boxed{1} & 5 & 2 & 3 & 4 \\ 4 & 18 & 2 & 8 & 12 \\ 3 & 11 & -6 & 1 & 4 \\ 0 & 2 & 6 & 4 & 4 \end{pmatrix} \quad \begin{array}{c} \textit{nothing to do} \\ \textit{(1. row)} \cdot (-4) + \textit{(2. row)} \\ \textit{(1. row)} \cdot (-3) + \textit{(3. row)} \\ \textit{nothing to do} \end{array}$$

$$\begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 0 & \boxed{-2} & -6 & -4 & -4 \\ 0 & -4 & -12 & -8 & -8 \\ 0 & 2 & 6 & 4 & 4 \end{pmatrix} \quad \begin{array}{c} \textit{(2. row)} \cdot (5/2) + \textit{(1. row)} \\ \textit{(2. row)} \cdot (-1/2) \\ \textit{(2. row)} \cdot (-2) + \textit{(3. row)} \\ \textit{(2. row)} + \textit{(3. row)} \end{array}$$

$$\begin{pmatrix} 1 & 0 & -13 & -7 & -6 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \\ \\ x_3 \textit{ arbitrary} \\ x_4 \textit{ arbitrary} \end{array}$$

$$\begin{aligned} x_1 - 13x_3 - 7x_4 &= -6 \quad \Longleftrightarrow \quad x_1 = 13x_3 + 7x_4 - 6 \\ x_2 + 3x_3 + 2x_4 &= 2 \quad \Longleftrightarrow \quad x_2 = -3x_3 - 2x_4 + 2 \end{aligned}$$

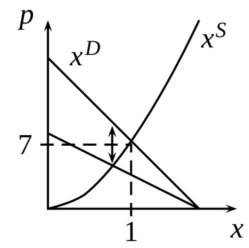
5.3.2 Cramer's Rule

Consider the system $\mathbf{Ax} = \mathbf{b}$ with a square matrix $\mathbf{A}$. This system of n quations and n unknowns has a unique solution if $\mathbf{A}$ is nonsingular. The solution is

$|\mathbf{A}| \neq 0$

$$x_1 = \frac{|\mathbf{D}_1|}{|\mathbf{A}|}, \quad \ldots, \quad x_n = \frac{|\mathbf{D}_n|}{|\mathbf{A}|},$$

where $\mathbf{D}_j$ results from replacing the j^{th} column in the matrix $\mathbf{A}$ by $\mathbf{b}$.



Example *Comparative statics of a market equilibrium*

market supply	$p = 5x^S + 2(x^S)^2$
market demand	$(1+t)p = 10 - 3x^D$
market equilibrium	$x^S = x^D$

Letting the consumer tax $t = 0$ *yields*

$$5x + 2x^2 = 10 - 3x \iff x^2 + 4x - 5 = 0 \implies \hat{x} = 1, \ \hat{p} = 7$$

What happens if the tax t *is increased by* $\mathrm{d}t$? *Evaluate total differentials to obtain an approximation!*

☞ *Taylor series*

$$\left.\begin{aligned} \mathrm{d}p &= 5\mathrm{d}x + 4x\mathrm{d}x \\ (1+t)\mathrm{d}p + p\mathrm{d}t &= -3\mathrm{d}x \end{aligned}\right\} \iff \begin{pmatrix} 1 & -5-4x \\ 1+t & 3 \end{pmatrix} \begin{pmatrix} \mathrm{d}p \\ \mathrm{d}x \end{pmatrix} = \begin{pmatrix} 0 \\ -p\,\mathrm{d}t \end{pmatrix}$$

Cramer's rule

$$\mathrm{d}p = \frac{\begin{vmatrix} 0 & -5-4x \\ -p\,\mathrm{d}t & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -5-4x \\ 1+t & 3 \end{vmatrix}} = \frac{-p(5+4x)\,\mathrm{d}t}{3+(1+t)(5+4x)} \xrightarrow{\hat{x}=1,\hat{p}=7,t=0} -\frac{21}{4}\,\mathrm{d}t$$

$$\mathrm{d}x = \frac{\begin{vmatrix} 1 & 0 \\ 1+t & -p\,\mathrm{d}t \end{vmatrix}}{\begin{vmatrix} 1 & -5-4x \\ 1+t & 3 \end{vmatrix}} = \frac{-p\,\mathrm{d}t}{3+(1+t)(5+4x)} \xrightarrow{\hat{x}=1,\hat{p}=7,t=0} -\frac{7}{12}\,\mathrm{d}t$$

6 Linear Programming

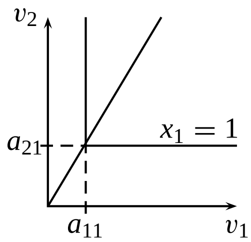
6.1 General Linear Programming Problems

Linear programming problems consist of a linear objective, or criterion, function and linear restrictions (equations or inequalities).

lineare Zielfunktion
lineare Nebenbedingungen

$$\begin{array}{rl} \max & p_1x_1 + p_2x_2 + \cdots + p_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \le b_n \\ & x_1, \quad x_2, \ldots, \quad x_n \ge 0 \end{array} \qquad \begin{array}{rl} \max & \mathbf{p}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\,\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \end{array}$$

Example (Leontief models)



$$x_1 = \min\left\{\frac{v_{11}}{a_{11}}, \frac{v_{21}}{a_{21}}\right\} \quad \text{and} \quad x_2 = \min\left\{\frac{v_{12}}{a_{12}}, \frac{v_{22}}{a_{22}}\right\}$$
$$v_1 = v_{11} + v_{12} \le b_1 \quad \text{and} \quad v_2 = v_{12} + v_{22} \le b_2$$

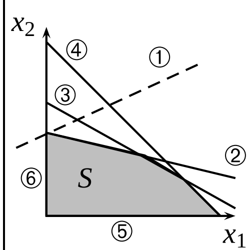
a_{ij} = input coefficient (minimum amount of v_i needed to produce one unit of x_j.)
Avoid any waste of resources!

$$v_{11} = a_{11}x_1,\; v_{21} = a_{21}x_1,\; v_{12} = a_{12}x_2,\; v_{22} = a_2x_2$$
$$\Longrightarrow v_1 = a_{11}x_1 + a_{12}x_2 \le b_1 \quad \text{and} \quad v_2 = a_{21}x_1 + a_{22}x_2 \le b_2$$

revenue maximization	$\max p_1x_1 + p_2\,x_2$
factor endowments	$a_{11}x_1 + a_{12}x_2 \le b_1$
	$a_{21}x_1 + a_{22}x_2 \le b_2$
non-negativity	$x_1, \quad x_2 \ge 0$

6.2 Graphic Solution Method

$$\begin{array}{rl} \max p_1x_1 + p_2\,x_2 & ① \\ a_{11}x_1 + a_{12}x_2 \le b_1 & ② \\ a_{21}x_1 + a_{22}x_2 \le b_2 & ③ \\ a_{31}x_1 + a_{32}x_2 \le b_3 & ④ \\ x_1, \quad x_2 \ge 0 & ⑤ \text{ and } ⑥ \end{array}$$



The (hopefully non-empty) feasible region S is a so-called convex polyhedron and corresponds to the shaded area in the picture. The corner points, or vertices, are called the extreme points of S. Given a linear objective function ①, any set of optimal solutions includes at least one such extreme point.

Ecken

6.3 Simplex Algorithm

The simplex algorithm starts at some known vertex $\mathbf{x}_0$ and switches to adjacent vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots$ until the algorithm stops. The final vertex is either a minimum solution or it indicates that the problem has no solution.

$\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$ vector of variables
$\mathbf{y} = (x_{n+1}, \ldots, x_{n+m})^\mathsf{T}$ vector of slack variables

Schlupfvariablen

$$\begin{array}{rcl} \max \mathbf{p}^\mathsf{T}\mathbf{x} & \xrightarrow{\mathbf{q}=-\mathbf{p}} & \min \mathbf{q}^\mathsf{T}\mathbf{x} \\ \mathbf{A}\,\mathbf{x} \leqq \mathbf{b} & \rightarrow & \mathbf{A}\,\mathbf{x} + \mathbf{y} = \mathbf{b} \\ \mathbf{x} \geqq \mathbf{0} & & \mathbf{x} \geqq \mathbf{0} \\ & & \mathbf{y} \geqq \mathbf{0} \end{array}$$

Suppose $\mathbf{b} \geqq \mathbf{0}$, then $\mathbf{x}_0 = \mathbf{0}$ (non-basis variables) and $\mathbf{y} = \mathbf{b}$ (basis variables) determine the first feasible vertex. The value of the objective function is $Q = \mathbf{q}^\mathsf{T}\mathbf{x}_0 = 0$.

non-basis variables

	·	② 1	2	⋯	n	·	·
basis variables	$n+1$	③ a_{11}	a_{12}	⋯	a_{1n}	④ b_1	⑦
	① ⋮	⋮	⋮	⋮	⋮	⋮	⋮
	$n+m$	a_{m1}	a_{m2}	⋯	a_{mn}	b_m	
	·	⑤ $-q_1$	$-q_2$	⋯	$-q_n$	⑥ Q	·

Step 1. Find a positive entry in the fields ⑤ and mark that column (pivot column). Stop if no such entry exists, the solution has been found.

Step 2. Find all positive elements of ③ in the pivot column (the problem has no solution of no such element exists) and note b_i/a_{ij} in the fields ⑦.

Step 3. Find the smallest quotient b_i/a_{ij} in ⑦ and mark that row (pivot row).

Step 4. The pivot row and the pivot column determine the pivot element in ③ – say a_{rs}. Now swap the index of the basis variable s with the non-basis variable r and compute the next tabular according to the following scheme.

- Substitute the pivot a_{rs} by $1/a_{rs}$.
- Divide the elements of the pivot row by a_{rs}.
- Divide the elements of the pivot column by $-a_{rs}$.
- Substitute all other elements d by $d - \frac{bc}{a}$, where

$$\begin{array}{ccc} a & \cdots & b \\ \vdots & & \vdots \\ c & \cdots & d \end{array}$$

with a being the pivot.

Step 5. Go to step 1 until the algorithm stops.

start tableau

	$\cdots$	i	$\cdots$	$\boxed{s}$	$\cdots$		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
k	$\cdots$	a_{ki}	$\cdots$	a_{ks}	$\cdots$	b_k	b_k/a_{ks}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\boxed{r}$	$\cdots$	a_{ri}	$\cdots$	$\boxed{a_{rs}}$	$\cdots$	b_r	b_r/a_{rs}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$\cdots$	$-q_i$	$\cdots$	$-q_s$	$\cdots$	Q	

end tableau

	$\cdots$	i	$\cdots$	$\boxed{r}$	$\cdots$		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
k	$\cdots$	$a_{ki} - \frac{a_{ri}a_{ks}}{a_{rs}}$	$\cdots$	$-\frac{a_{ks}}{a_{rs}}$	$\cdots$	$b_k - \frac{b_r a_{ks}}{a_{rs}}$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$\boxed{s}$	$\cdots$	$\frac{a_{ri}}{a_{rs}}$	$\cdots$	$\frac{1}{a_{rs}}$	$\cdots$	$\frac{b_r}{a_{rs}}$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
	$\cdots$	$-q_i + \frac{a_{ri}q_s}{a_{rs}}$	$\cdots$	$\frac{q_s}{a_{rs}}$	$\cdots$	$Q + \frac{b_r q_s}{a_{rs}}$	

Example

$$\begin{aligned}
&\max 250x_1 + 45\,x_2 \\
&x_1 \le 50 \\
&x_2 \le 200 \\
&x_1 + 0.2x_2 \le 72 \\
&150x_1 + 25\,x_2 \le 10000 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}
\qquad
\begin{aligned}
&\min -250x_1 - 45\,x_2 \\
&x_1 + x_3 = 50 \\
&x_2 + x_4 = 200 \\
&x_1 + 0.2x_2 + x_5 = 72 \\
&150x_1 + 25\,x_2 + x_6 = 10000 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}$$

①	x_1	x_2		
x_3	$\boxed{1}$	0	50	$\boxed{50}$
x_4	0	1	200	–
x_5	1	0.2	72	72
x_6	150	25	10000	66.67
	$\boxed{250}$	45	0	

②	x_3	x_2		
x_1	1	0	50	–
x_4	0	1	200	200
x_5	-1	0.2	22	110
x_6	-150	$\boxed{25}$	2500	$\boxed{100}$
	-250	$\boxed{45}$	-12500	

③	x_3	x_6		
x_1	1	0	50	50
x_4	6	-0.04	100	16.67
x_5	$\boxed{0.2}$	-0.008	2	$\boxed{10}$
x_2	-6	0.04	100	–
	$\boxed{20}$	-1.8	-17000	

④	x_5	x_6		
x_1	-5	0.04	40	
x_4	-30	0.2	40	
x_3	5	-0.04	10	
x_2	30	-0.2	160	
	-100	-1	-17200	

Solution $x_1 = 40$, $x_2 = 160$, $Q = -17200$ *(17200 for the maximum problem)*

6.4 Duality

Each *primal* linear program (P) is assigned a *dual* linear program (D).

primal problem (P)
dual problem (D)

$$\begin{array}{ll} \min & p_1x_1 + p_2x_2 + p_3x_3 \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \geq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ & a_{31}x_2 + a_{32}x_2 + a_{33}x_3 \leq b_3 \\ & x_1 \geq 0 \\ & x_2 \text{ unrestr.} \\ & x_3 \leq 0 \end{array} \qquad \begin{array}{ll} \max & b_1u_1 + b_2u_2 + b_3u_3 \\ \text{s.t.} & a_{11}u_1 + a_{21}u_2 + a_{31}u_3 \leq p_1 \\ & a_{12}u_1 + a_{22}u_2 + a_{32}u_3 = p_2 \\ & a_{13}u_1 + a_{23}u_2 + a_{33}u_3 \geq p_3 \\ & u_1 \geq 0 \\ & u_2 \text{ unrestr.} \\ & u_3 \leq 0 \end{array}$$

The rules for forming the dual are given by

Primal Problem (P)	Dual Problem (D)
minimize primal objective	maximize dual objective
objective coefficients	RHS of dual restrictions
RHS of primal restrictions	objective coefficients
coefficient matrix	transposed coefficient matrix
primal relation	dual variable ($i = 1, \ldots, m$)
(ith) inequality $\geq$	$u_i \geq 0$
(ith) inequality $\leq$	$u_i \leq 0$
(ith) equation $=$	u_i unrestricted in sign
primal variable ($j = 1, \ldots, n$)	dual relation
$x_j \geq 0$	(jth) inequality $\leq$
$x_j \leq 0$	(jth) inequality $\geq$
x_j unrestricted in sign	(jth) equation $=$

Unbounded-infeasible relationship If (P) is unbounded, then (D) is infeasible, and vice versa.
Weak duality theorem If $\mathbf{x}$ is a feasible vector of the primal problem (P) and $\mathbf{u}$ a feasible vector of the dual problem (D), then

$$\mathbf{p}^\top\mathbf{x} \geq \mathbf{b}^\top\mathbf{u}\,.$$

Strong duality theorem A feasible vector $\hat{\mathbf{x}}$ of the primal problem (P) is a minimum solution to (P) if and only if a feasible vector $\hat{\mathbf{u}}$ of (D) exists with

$$\mathbf{p}^\top\hat{\mathbf{x}} = \mathbf{b}^\top\hat{\mathbf{u}}.$$

In this case $\hat{\mathbf{u}}$ is a maximum solution of the dual problem (D). A similar statement results if we start with (D) instead of (P).
Complementary slackness theorem A feasible vector $\hat{\mathbf{x}}$ of the primal problem (P) is a minimum solution to (P) if and only if a feasible vector $\hat{\mathbf{u}}$ of (D) exists with

$$\begin{array}{ll} (a_{11}\hat{x}_1 + a_{12}\hat{x}_2 + a_{13}\hat{x}_3 - b_1)\hat{u}_1 = 0; & (a_{11}\hat{u}_1 + a_{21}\hat{u}_2 + a_{31}\hat{u}_3 - p_1)\hat{x}_1 = 0 \\ a_{21}\hat{x}_1 + a_{22}\hat{x}_2 + a_{23}\hat{x}_3 - b_2 = 0; & a_{12}\hat{u}_1 + a_{22}\hat{u}_2 + a_{32}\hat{u}_3 - p_2 = 0 \\ (a_{31}\hat{x}_1 + a_{32}\hat{x}_2 + a_{33}\hat{x}_3 - b_3)\hat{u}_3 = 0; & (a_{13}\hat{u}_1 + a_{23}\hat{u}_2 + a_{33}\hat{u}_3 - p_3)\hat{x}_3 = 0 \end{array}$$

7 Algebra

7.1 Operations with Sets

union	$\mathcal{A} \cup \mathcal{B} = \{x \mid x \in \mathcal{A} \vee x \in \mathcal{B}\}$ (elements are members of $\mathcal{A}$ *or* $\mathcal{B}$)	*Vereinigungsmenge*
intersection	$\mathcal{A} \cap \mathcal{B} = \{x \mid x \in \mathcal{A} \wedge x \in \mathcal{B}\}$ (elements are members of $\mathcal{A}$ *and* $\mathcal{B}$)	*Schnittmenge*
disjoint sets	$\mathcal{A} \cap \mathcal{B} = \emptyset$ ($\mathcal{A}$ and $\mathcal{B}$ have no elements in common)	*disjunkte Mengen*
set difference	$\mathcal{A} - \mathcal{B} \equiv \mathcal{A} \setminus \mathcal{B} = \{x \mid x \in \mathcal{A} \wedge x \notin \mathcal{B}\}$ (all elements of $\mathcal{A}$ which are not in $\mathcal{B}$)	*Differenzmenge*
complement	$C(\mathcal{A}) \equiv \overline{A} = \{x \mid x \notin \mathcal{A}\}$	*Komplement*
subset	$\mathcal{A} \subset \mathcal{B} \iff \mathcal{A}$ is a subset of $\mathcal{B}$	*Teilmenge*
superset	$\mathcal{A} \supset \mathcal{B} \iff \mathcal{A}$ is a superset of $\mathcal{B}$	*Obermenge*
equality	$\mathcal{A} = \mathcal{B} \iff \mathcal{A}$ and $\mathcal{B}$ include the same elements	

7.2 Basic Laws of Set Algebra

commutative law	$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ and $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$	*Kommutativgesetz*
associative law	$(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$ and $(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$	*Assoziativgesetz*
distributive law	$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$ and $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$	*Distributivgesetz*
absorption law	$\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$ and $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$	*Absorptionsgesetz*
idempotence law	$\mathcal{A} \cap \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$	*Idempotenzgesetz*
de Morgan's rule	$\overline{\mathcal{A} \cap \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$ and $\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$	*de Morgansche Regel*
further laws	$\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset, \quad \mathcal{A} \cap \emptyset = \mathcal{A}, \quad \mathcal{A} \cup \emptyset = \mathcal{A}, \quad \overline{\overline{\mathcal{A}}} = \mathcal{A}$	

7.3 Binary Relations

A *binary relation* R on a set $\mathcal{A}$ denotes a determined relationship between two not necessarily distinct element of $\mathcal{A}$ (e.g., a is better, lighter, or not smaller than b). A binary relation can have different properties: *binäre Relation*

reflexivity	$aRa \quad \forall\, a \in \mathcal{A}$	*Reflexivität*
irreflexivity	$\neg(aRa) \quad \forall\, a \in \mathcal{A}$	*Irreflexivität*
symmetry	$aRb \implies bRa \quad \forall a, b \in \mathcal{A}$	*Symmetrie*
anti-symmetry	$aRb \wedge bRa \implies a = b \quad \forall\, a, b \in \mathcal{A}$	*Antisymmetrie*

transitivity	$aRb \wedge bRc \implies aRc \quad \forall\, a, b, c \in \mathcal{A}$	*Transitivität*
linearity	$(aRb \vee bRa) \quad \forall\, a, b \in \mathcal{A}$	*Vollständigkeit*

A binary relation R is called *ordering* if R is reflexive, anti-symmetrical, and transitive. *Ordnung*

Examples *Inequalities (e.g., greater than $>$ or not smaller than $\geq$) are binary relations. Preferences can be described by binary relations: a commodity bundle* $\mathbf{x}$ *is better than* $\mathbf{x}'$ *is denoted by* $\mathbf{x} \succ \mathbf{x}'$. *Indifference between* $\mathbf{x}$ *and* $\mathbf{x}'$ *is written as* $\mathbf{x} \sim \mathbf{x}'$. *The relation* $\mathbf{x} \succcurlyeq \mathbf{x}'$ *says that* $\mathbf{x}$ *in not worse than* $\mathbf{x}'$.
(Note: $\mathbf{x} \succcurlyeq \mathbf{x}' \implies \mathbf{x} \succ \mathbf{x}' \vee \mathbf{x} \sim \mathbf{x}'$*)*

Examples *Suppose that preferences* $\succcurlyeq$ *are ordered and can be represented by a real valued function* u *on* $\mathcal{X}$ *(order preserving mapping) as*

$$\text{(1)} \qquad \mathbf{x} \succ \mathbf{x}' \implies u(\mathbf{x}) > u(\mathbf{x}') \qquad \forall\, \mathbf{x}, \mathbf{x}' \in \mathcal{X}.$$

Then, u *is called an* ordinal utility function. *It is unique up to any monotone transformation such as computing* $\ln u(\mathbf{x})$. *ordinale Nutzenfunktion*

A cardinal utility function u *satisfies (1) and is unique up to a positive* $(\alpha > 0)$ *linear transformation of the form* $\tilde{u}(\mathbf{x}) = \alpha u(\mathbf{x}) + \beta$. *The two functions* $\tilde{u}$ *and* u *differ only by their origin and units. The new units apply not only to utility values but also to utility differences, i.e.,* $u(\mathbf{x}) - u(\mathbf{x}') \to \alpha[u(\mathbf{x}) - u(\mathbf{x}')]$. *kardinale Nutzenfunktion*

8 Probability Theory and Statistical Distributions

8.1 Combinatorial Calculus

Kombinatorik

cardinality	$\#\mathcal{S} = n =$ number of elements of $\mathcal{S}$	*Mächtigkeit* *Kardinalität*
factorial	$n! = 1 \cdot 2 \cdots (n-1) \cdot n$ $\quad (0! = 1)$	*Fakultät*
binomial coefficient	$\binom{n}{m} = \frac{n!}{m!(n-m)!}$	*Binom*
multinomial coefficient	$\frac{n!}{n_1!\, n_2! \cdots n_k!}$ where $n = n_1 + n_2 + \cdots + n_k$	
permutation	n distinct elements: $n!$ $\quad abc, acb, bac, bca, cab, cba$ n elements, k classes with $n_1, n_2, \ldots n_k$ elements $\frac{n!}{n_1!\, n_2! \cdots n_k!}$ $\quad aab, aba, baa$	

	without recurrence	with recurrence
variations (with order)	$k!\binom{n}{k} = \frac{n!}{(n-k!)}$ ab, ba, ac, ca, bc, cb	n^k $aa, ba, ca, ab, bb, cb, ac, bc, cc$
combinations (without order)	$\binom{n}{k}$ ab, ac, bc	$\binom{n+k-1}{k}$ aa, ba, ca, bb, cb, cc

variation	= the order of elements is considered	$(ab \neq ba)$
combinations	= the order of elements is not considered	$(ab = ba)$
without recurrence	= each element (a, b, c) in a group is unique	

Examples *6 numbers out of 45* $\rightarrow \binom{45}{6}$*; 3 right and 3 blanks* $\rightarrow \binom{6}{3}\binom{39}{3}$*; 9 games with victory, defeat or draw* $\rightarrow 3^9$*; 6 people having birthday in exactly two month* $\rightarrow (2^6 - 2)\binom{12}{6}$*;*

8.2 Random Variables

8.2.1 Probability of Events

outcome — the result of an experiment (e.g. rolling a die); the set of all possible outcomes of a probability experiment is called a *sample space* (e.g. $\Omega = \{1, 2, 3, 4, 5, 6\}$). *Ereignismenge*

event — any collection of outcomes of an experiment (i.e. any subset of Ω); any event which consists of a single outcome in the sample space is called an *elementary* or *simple event*; events which consist of more than one outcome are called *compound events* (e.g. $\mathcal{A} = \{2, 4, 6\}$). *Elementarereignis*

probability	$P(\mathcal{A})$ denotes the probability that an event (success) $\mathcal{A} \subseteq \Omega$ occurs. $0 \leq P(\mathcal{A}) \leq 1$	*Wahrscheinlichkeit*
certain event	$P(\Omega) = 1$	
impossible event	$P(\emptyset) = 0$	
mutually exclusive events	$\mathcal{A} \cap \mathcal{B} = \emptyset \implies P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B})$	
complement	$P(\overline{\mathcal{A}}) = 1 - P(\mathcal{A})$ denotes the probability of a failure $\overline{\mathcal{A}}$ — i.e. the complement of $\mathcal{A}$.	$\mathcal{A} \cup \overline{\mathcal{A}} = \Omega$ $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$
addition rule	$P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A} \cap \mathcal{B})$	
conditional probability of $\mathcal{A}$ given $\mathcal{B}$	$P(\mathcal{A}\mid\mathcal{B}) = \dfrac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})}$	$P(\mathcal{B}) \neq 0$
independent events	$P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A})\, P(\mathcal{B})$ and $P(\mathcal{A}\mid\mathcal{B}) = P(\mathcal{A})$	
law of total probability	$P(\mathcal{A}) = P(\mathcal{A}\mid\mathcal{B})\, P(\mathcal{B}) + P(\mathcal{A}\mid\overline{\mathcal{B}})\, P(\overline{\mathcal{B}})$	
Bayes' theorem	$P(\mathcal{A}\mid\mathcal{B}) = \dfrac{P(\mathcal{B}\mid\mathcal{A})\, P(\mathcal{A})}{P(\mathcal{B}\mid\mathcal{A})\, P(\mathcal{A}) + P(\mathcal{B}\mid\overline{\mathcal{A}})\, P(\overline{\mathcal{A}})}$	

8.2.2 Discrete Random Variables

Let X be a random variable, then $P\{X = x\}$ is the probability that X takes the value x. Other events are $\{X \leq x\}$ or $\{a < X \leq b\}$.

notation $P\{X = x\} \equiv P(x)$

Probability mass function (pmf) The pmf of a *discrete random variable* X is determined by the collection of numbers $\{\pi_i\}$ satisfying $P\{X = x_i\} = \pi_i \geq 0$ for all i and $\sum_i p_i = 1$.

Wahrscheinlichkeitsfunktion $P(x_i) = \pi_i$

expected value or mean	$\mu = E(X) = \sum_i \pi_i\, x_i$	*Erwartungswert*
variance	$V(X) = E(X - \mu) = \sum_i (x_i - \mu)^2 \pi_i$ $= E(X^2) - (E(X))^2 = \sum_i x_i^2\, \pi_i - \mu^2$	*Varianz*
standard deviation	$\sigma = \sqrt{V(X)}$	*Standardabweichung*

8.2.3 Continuous Random Variables

(Cumulative) distribution function (cdf) The cdf of a *continuous random variable* X is a nondecreasing, right continuous function F which satisfies $F(x) = P\{X \leq x\}$, $F(-\infty) = 0$, and $F(+\infty) = 1$. Bear in mind that $P\{X = x\} = 0$ for any continuous random variable X.

Verteilungsfunktion

Probability density function (pdf) The pdf results from

Dichtefunktion

$$f(x) = \lim_{dx \downarrow 0} \frac{F(x + dx) - F(x)}{dx} \qquad \text{or} \qquad f(x) = F'(x),$$

provided that F is differentiable.

expected value or mean	$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$	
variance	$V(X) = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\, \mathrm{d}x$	

8.3 Probability Distributions

8.3.1 Discrete Random Variables

binomial distribution with parameters $0 < p < 1$ and $n \in \mathbb{N}$

pmf	$P\{X = x\} = f(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, 1, \ldots, n$
mean	$E(X) = np$	
variance	$V(X) = np(1-p)$	

hypergeometric distibution with parameters $M, N, n \in \mathbb{N}$, $n \leq N$, $M < N$

pmf	$P\{X = x\} = f(x; M, N, n) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	
mean	$E(X) = np \quad \text{with } p = M/N$	
variance	$V(X) = np(1-p)(N-n)/(N-1)$	

Poisson distribution with parameter $\mu > 0$

pmf	$P\{X = x\} = f(x; \mu) = \dfrac{\mu^x \mathrm{e}^{-\mu}}{x!}$	$x = 0, 1, 2 \ldots$
mean	$E(X) = \mu$	
variance	$V(X) = \mu$	

The sum Σ of n independent Poisson random variables X_i with parameter μ_i is also Poisson distributed with parameter $\mu = \mu_1 + \cdots + \mu_n$.

8.3.2 Continuous Random Variables

normal distribution with parameters μ and $\sigma > 0$

pdf	$f(x; \mu, \sigma) = \dfrac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mean	$E(X) = \mu$
variance	$V(X) = \sigma^2$

standardized normal distribution: $\mu = 0$ and $\sigma = 1$

exponential distribution with parameter $\mu > 0$

pdf	$f_{Exp}(x;\mu) = \dfrac{e^{-x/\mu}}{\mu}$	$x > 0$
cdf	$F_{Exp}(x;\mu) = 1 - e^{-x/\mu}$	$x \geq 0$
mean	$E(X) = \mu$	
variance	$V(X) = \mu^2$	

The exponential distribution is a special case of the Gamma distribution when $\alpha = 1$ and $\beta = \mu$.
An exponentially distributed random variable has *no memory (Markov property)*.

$$P\{X > x + \Delta |\, X > x\} = P\{X > \Delta\} = e^{-\Delta/\mu}$$

uniform rectangular distribution with parameters α and β

pdf	$f(x;\alpha,\beta) = \dfrac{1}{\beta - \alpha}$
mean	$E(X) = (\alpha + \beta)/2$
variance	$V(X) = (\beta - \alpha)^2/12$

geometric distribution with parameter p

pdf	$f(x;p) = p(1-p)^x$
mean	$E(X) = (1-p)/p$
variance	$V(X) = (1-p)/p^2$

Gamma Function The Gamma function Γ has been introduced by Euler to calculate the *factorial* of positiv real numbers.

$$\Gamma(k) = \int_0^\infty u^{k-1}\, e^{-u} du \quad \text{with} \quad k > 0,$$

recursion formula $\Gamma(k+1) = k!$ provided $k > 0$ is an integer

Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$

pdf	$f_{Gam}(x;\alpha,\beta) = \dfrac{(x/\beta)^{\alpha-1}\, e^{-x/\beta}}{\beta\,\Gamma(\alpha)}$	$x > 0$
cdf	$F_{Gam}(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \int_0^{x/\beta} u^{\alpha-1} e^{-u} du$	$x \geq 0$
mean	$E(X) = \alpha\beta$	
variance	$V(X) = \alpha\beta^2$	

The sum Σ of n independent Gamma distributed random variables X_i with parameters α_i and β is Gamma distributed with parameters $\alpha = \alpha_1 + \cdots + \alpha_n$ and β.

χ^2-distribution with $a \in \mathbb{N}$ degrees of freedom

pdf	$f_{\chi^2}(x;a) = \dfrac{x^{a/2-1}\,\mathrm{e}^{-x/2}}{2^{a/2}\Gamma(a/2)}$	$x \geq 0$
mean	$E(X) = a$	
variance	$V(X) = 2a$	

The sum Σ of n independent χ^2 distributed random variables X_i with parameters a_i is χ^2 distributed with parameter $a = a_1 + \cdots + a_n$.

t-distribution with $a \in \mathbb{N}$ degrees of freedom

pdf	$f_t(x;a) = \dfrac{\Gamma(\frac{a+1}{2})}{\sqrt{a\pi}\,\Gamma(a/2)} \left(1 + \dfrac{x^2}{a}\right)^{-(a+1)/a}$	$x \in \mathbb{R}$
mean	$E(X) = 0 \quad \text{if } a \geq 2$	
variance	$V(X) = \dfrac{a}{a-2} \quad \text{if } a \geq 3$	

F-distribution with $a_1, a_2 \in \mathbb{N}$ (degrees of freedom)

pdf	$f_F(x;a_1,a_2) = \dfrac{\Gamma(\frac{a_1+a_2}{2})}{\Gamma(a_1/a_2)\,\Gamma(a_2/a_1)} \left(\dfrac{a_1}{a_2}\right)^{a_1/2} \times \dfrac{x^{a_1/2-1}}{(1 + x\,a_1/a_2)^{(a_1+a_2)/2}}$	$x \geq 0$
mean	$E(X) = \dfrac{a_2}{a_2-2} \quad \text{if } a_2 > 2$	
variance	$V(X) = \dfrac{2a_2^2(a_1+a_2-2)}{a_1(a_2-2)^2(a_2-4)} \quad \text{if } a_2 > 4$	

Beta-distribution with $\alpha > 0$, $\beta > 0$

pdf	$f_B(x;\alpha,\beta) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$	$0 \leq x \leq 1$
Beta function	$B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u = \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	
mean	$E(X) = \dfrac{\alpha}{\alpha+\beta}$	
variance	$V(X) = \dfrac{\alpha\,\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

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